More conditional probability.

Two topics for today:

1. Independent events
2. Bayes' formula.

1) We say events E and F are <u>independent</u> if $P(E) = P(E|F)$ (that is, E is just as likely whether or not F occurs). In this case, the multiplication rule

$$P(EF) = P(F)P(E|F)$$

becomes

$$\boxed{P(EF) = P(E)P(F)} \qquad (4.1)$$

Multiplication rule for <u>independent</u> events

<u>Example 1.</u>

Two unfair dice are rolled. On Die 1, odd numbers are twice as likely as evens; on Die 2, evens are 3x as likely as odds. Find $P(\text{sum on dice is odd})$.

<u>Solution.</u>

Let E_1 be the event that the first die is even. Similarly define E_2, O_1, O_2. Let S be the event that the sum is odd. Then

$$S = E_1O_2 \cup E_2O_1,$$

since odd+even = even+odd = odd but even+even = odd+odd = even.

By independence of the dice,

$$P(S) = P(E_1)P(O_2) + P(E_2)P(O_1).$$

Now $P(O_1) = 2P(E_1)$ and $P(O_1) + P(E_1) = 1$,
so $P(O_1) = 2/3$ and $P(E_1) = 1/3$.
Similarly, $P(E_2) = 3/4$, $P(O_2) = 1/4$.

So
$$P(S) = \tfrac{1}{3}\cdot\tfrac{1}{4} + \tfrac{2}{3}\cdot\tfrac{3}{4} = \frac{1+6}{12} = \frac{7}{12} = 0.5833.$$

Remark. (4.1) generalizes: if $E_1, E_2, \ldots$ are all independent, then

$$\boxed{P(E_1E_2E_3\cdots E_n) = P(E_1)P(E_2)\cdots P(E_n)}$$

2) Let E, F be events. To say E happens is to say either EF happens or EF^c happens. So
$$E = EF \cup EF^c.$$
Since EF and EF^c are mutually exclusive,

$$P(E) = P(EF) + P(EF^c).$$

So, by the multiplication rule,

$$\boxed{P(E) = P(F)P(E|F) + P(F^c)P(E|F^c).} \qquad (3.1)$$

Bayes's Formula

Example 2.
A box contains 3 coins: 2 are fair, and one has $P(H) = 1/4$.

A coin is chosen at random from the box and flipped twice. What's the probability it lands heads both times?

Solution.

B : both flips land heads.

F : a fair coin is chosen.

Then

$$P(B) = P(F)P(B|F) + P(F^c)P(B|F^c)$$
$$= \tfrac{2}{3} \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} + \tfrac{1}{3} \cdot \tfrac{1}{4} \cdot \tfrac{1}{4}$$
$$= \tfrac{2}{12} + \tfrac{1}{48} = 0.1875.$$

A generalization of (3.1):

If $F_1, F_2, F_3, \ldots, F_n$ are mutually exclusive and exhaustive (together they constitute the whole sample space), then

$$\boxed{P(E) = P(F_1)P(E|F_1) + P(F_2)P(E|F_2) + \ldots + P(F_n)P(E|F_n).}$$

Example 3.

A game show features a prize in a locked box, and 8 keys, only 3 of which open the box. A contestant rolls a fair, six-sided die: they get one key (at random) if (A) a 1, 2, or 3 shows, 2 keys if (B) a 4 or 5 shows, and 3 if (C) a 6 shows. What is P(win the prize)?

Solution. It's easier to compute P(lose). Since A, B, C are mutually exclusive and exhaustive, we have

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$$P(\text{lose}) = P(A)P(\text{lose}\mid A) + P(B)P(\text{lose}\mid B) + P(C)P(\text{lose}\mid C)$$

$$= \frac{1}{2}\cdot\frac{5}{8} + \frac{1}{3}\cdot\frac{5\cdot 4}{8\cdot 7} + \frac{1}{6}\cdot\frac{5\cdot 4\cdot 3}{8\cdot 7\cdot 6} = 0.46131.$$

So $P(\text{win}) = 1 - 0.46131 = 0.53869.$