Conditional probability.

Notation: if E and F are events, we write $P(E|F)$ ("P of E given F") for the probability of E happening assuming F happens too.

1) A general formula for $P(E|F)$.
$P(E|F)$ should measure, roughly, "relative to the likelihood of F, how likely is it that E happens too?"

So we define

$$\boxed{P(E|F) = \frac{P(EF)}{P(F)}.} \qquad (2.1)$$

Definition of Conditional Probability

Example 1.

45% of Boulder residents have cool shoes. 35% have cool shoes but don't like zucchini.

Given that a Boulder resident has cool shoes, find the probability that they like zucchini.

Solution. Define events

C: has cool shoes, Z: likes zucchini.

Then we're looking for

$$P(Z|C) = \frac{P(CZ)}{P(C)}.$$

Now if you have cool shoes, you either like zucchini or not. So

$$C = CZ \cup (C-Z).$$

So

$$P(C) = P(CZ) + P(C-Z).$$

Since $P(C) = .45$ and $P(C-Z) = .35$, we have $P(CZ) = .1$. So

$$P(Z|C) = \frac{P(CZ)}{P(C)} = \frac{.1}{.45} = 0.2222\ldots$$
$$\sim 22.22\%.$$

2) <u>Equally likely case.</u>

Note: if all outcomes in a sample space S are equally likely, then

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{|EF|/|S|}{|F|/|S|}.$$

The $|S|$'s cancel, so:

If all events in a sample space are equally likely, then

$$P(E|F) = \frac{|EF|}{|F|} \text{ for any events } E, F \subseteq S.$$

<u>Example 2.</u>

Two fair dice are rolled; you're told that one landed on 6. What's the probability that both did?

Solution.

$P(\text{both are } 6 \mid \text{one is } 6)$

$$= \frac{|\text{both are } 6|}{|\text{one is } 6|}$$

$$= \frac{|\{6,6\}|}{|\{16, 26, 36, 46, 56, 66, 65, 64, 63, 62, 61\}|}$$

$$= \frac{1}{11} \approx 9.09\%.$$

3) The multiplication rule.

If we solve equation (2.1) for $P(EF)$, we get

$$\boxed{P(EF) = P(F)\,P(E \mid F)} \qquad (2.2)$$

The multiplication rule

Example 3.

Two 5-card hands are dealt, in succession, from a standard deck.

Find $P(\text{neither hand has an ace})$.

Solution. Define

A_1: first hand has no ace

A_2: second hand has no ace.

Then $P(A_1 A_2) = P(A_1)\,P(A_2 \mid A_1)$.

For A_1, with 52 original cards, we must draw 5 out of 48 non-aces.

So
$$P(A_1) = \binom{48}{5} \Big/ \binom{52}{5}.$$

Next: assuming A_1, 47 cards remain $(52-5=47)$, from which we must draw 5 of 43 non-aces. $(52-5-4=43.)$

So $P(A_2 | A_1) = \binom{43}{5} \Big/ \binom{47}{5}.$

So
$$P(A_1 A_2) = \frac{\binom{48}{5}}{\binom{52}{5}} \cdot \frac{\binom{43}{5}}{\binom{47}{5}} = 0.413445.$$