Equally likely outcomes.

Notation: given any set E, let $|E|$ denote the number of elements in E.

FACT (which can be derived from the axioms):

> if every event in a sample space S is equally likely, then for any event E in S,
>
> $$P(E) = \frac{|E|}{|S|} \quad \text{(assuming } |S| \text{ is finite).}$$

Example 1.
Roll two dice; write $S = \{ij : 1 \le i, j \le 6\}$. Then $|S| = 36$. So for example

$$P(\text{sum is } 3) = \frac{|\text{sum is } 3|}{36} = \frac{|\{13, 22, 31\}|}{36} = \frac{3}{36} = .0833,$$

$$P(\text{numbers are at least 3 apart}) = \frac{|\{14, 15, 16, 25, 26, 36, 41, 51, 52, 61, 62, 63\}|}{36}$$

$$= \frac{12}{36} = .3333, \quad \text{etc.}$$

Example 2. (See HW 3, Problem 15.)
If all 5-card poker hands are equally likely, find
(b) $P(\text{one pair})$; (c) $P(\text{two pair})$.

Solution.
(b) First choose the face value (denomination)

for the pair: 13 choices. Then choose two cards of this value: $\binom{4}{2}$ ways. Now choose remaining 3 face values: $\binom{12}{3}$ ways. Finally, for each of these last 3 values, choose one card: 4 ways for each value. So

$$P(\text{one pair}) = \frac{13\cdot\binom{4}{2}\cdot\binom{12}{3}\cdot 4\cdot 4\cdot 4}{\binom{52}{5}} = 0.422569.$$

(c) Similarly,

$$P(\text{two pair}) = \frac{\overbrace{\binom{13}{2}}^{\text{choose face values for the two pairs}}\overbrace{\binom{4}{2}\binom{4}{2}}^{\text{choose 2 cards for each pair}}\cdot\overbrace{44}^{\text{choose last card}}}{\binom{52}{5}} = 0.047539.$$

<u>Example 3</u> (see HW 3 #16).

5 fair, 6-sided dice are rolled. Find

(d) P(3 alike).

<u>Solution.</u>

Think of an outcome as a string $ijk\ell m$, where $1 \le i, j, k, \ell, m \le 6$. Then there are 6^5 outcomes.

(d) Choose a number shared by the 3 alike: 6 choices. Choose which 3 of the 5 dice have this number: $\binom{5}{3}$ choices. The remaining two dice must differ from each other and from the 3 alike: $5\cdot 4$ choices. So

$$P(3\text{ alike}) = \frac{6\cdot\binom{5}{3}\cdot 5\cdot 4}{6^5} = 0.154321.$$

Example 4 (see HW 3 #12).

Consider events S, F, G in a sample space V, with
$|V| = 100$, $|S| = 28$, $|F| = 26$, $|G| = 16$, $|SF| = 12$,
$|SG| = 4$, $|FG| = 6$, $|SFG| = 2$.

(a) Find $P(S^c F^c G^c)$.
(c) Find P(if two outcomes are chosen at random, at least one is in $S \cup F \cup G$).

Solution

(a) Note that being in neither S, F, or G is the complement of being in at least one. So

$$P(S^c F^c G^c) = 1 - P(S \cup F \cup G) \quad \text{(by Prop. 4.1)}$$
$$= 1 - (P(S) + P(F) + P(G) - P(SF) - P(SG) - P(FG) + P(SFG)) \quad \text{(by Prop. 4.4)}$$
$$= 1 - (.28 + .26 + .16 - .12 - .04 - .06 + .02)$$
$$= .5.$$

(c) Let A be the event that the first outcome chosen is in $E \cup F \cup G$, and B the outcome that the second is.

We have
$$P(A \cup B) = P(A) + P(B) - P(AB).$$
We know $P(A) = P(B) = P(S \cup F \cup G) = .5$, by part (a).

What is $P(AB)$? Well, there are $\binom{100}{2}$ ways of choosing 2 outcomes from the 100, and $\binom{50}{2}$ ways of choosing 2 outcomes in $\underline{S \cup F \cup G}$ from the 50. So

$$P(AB) = \binom{50}{2} \Big/ \binom{100}{2}.$$

So

$$P(A \cup B) = P(A) + P(B) - P(AB)$$
$$= .5 + .5 - \frac{\binom{50}{2}}{\binom{100}{2}} = .752525.$$