More fun with counting:

A) Counting subsets.

Example 1.

Use two methods to count the subsets of $\{a,b,c,d\}$ that have r elements, for $0 \le r \le 4$.

Solution

Method 1: write them down.

$r=0$: $\emptyset$. 1 subset.

$r=1$: $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$. 4 subsets.

$r=2$: $\{a,b\}$, $\{a,c\}$, $\{a,d\}$, $\{b,c\}$, $\{b,d\}$, $\{c,d\}$. 6 subsets.

$r=3$: $\{a,b,c\}$, $\{a,b,d\}$, $\{a,c,d\}$, $\{b,c,d\}$: 4 subsets.

$r=4$: (DIY) 1 subset.

Method 2: An r-element subset is a choice of r elements out of the four. There are $\binom{n}{r}$ such choices, and therefore $\binom{n}{r}$ such subsets. We compute:

$$\binom{4}{0} = \frac{4!}{0!\,4!} = \frac{1}{0!} = \frac{1}{1} = 1 \quad \text{(we define } 0! = 1\text{)};$$

$$\binom{4}{1} = \frac{4!}{1!\,3!} = 4, \quad \binom{4}{2} = \frac{4!}{2!\,2!} = \frac{4\cdot 3}{2\cdot 1} = 6,$$

$$\binom{4}{3} = \frac{4!}{3!\,1!} = 4, \quad \binom{4}{4} = \frac{4!}{4!\,0!} = 1,$$

agreeing with Method 1.

Remark:

note that $1+4+6+4+1 = 16 = 2^4$. In general, a set with n elements has 2^n subsets (if you include $\emptyset$ as a subset).

B) Multinomial coefficients.

Example: 30 people go out for ice cream. In how many ways can 14 of them have Rocky Road (RR), 7 have bubble gum (BG), 4 have cookie dough (CD), and 5 have Cheetos Flamin' Hot (CFH)?

Solution

There are $\binom{30}{14}$ choices for which 14 get RR. Once chosen, there are $\binom{16}{7}$ options for who gets BG, then $\binom{9}{4}$ for CD, and finally $\binom{5}{5}$ for CFH. All told, there are

$$\binom{30}{14}\cdot\binom{16}{7}\cdot\binom{9}{4}\cdot\binom{5}{5}$$

$$= \frac{30!}{14!\,\cancel{16!}}\cdot\frac{\cancel{16!}}{7!\,\cancel{9!}}\cdot\frac{\cancel{9!}}{4!\,\cancel{5!}}\cdot\frac{\cancel{5!}}{5!\,0!}$$

$$= \frac{30!}{14!\,7!\,4!\,5!} \quad \text{ways}$$

In general:

The number of ways of dividing n distinct objects into r distinct groups, of sizes $n_1, n_2, n_3, \ldots, n_r$ (with $n_1+n_2+\cdots+n_r=n$), is

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}, \text{ also denoted } \binom{n}{n_1, n_2, \ldots, n_r} *$$

*called a multinomial coefficient.

Remark:
If the groups are indistinguishable, the number is smaller.

Example:
In how many ways can 30 people be divided into 5 groups of 6 (with no group being distinguished from another)?

Solution
If we distinguish groups, the answer is

$$\binom{30}{6,6,6,6,6}.$$

But we don't, so we divide by 5! (the number of ways of arranging the groups), to get

$$\binom{30}{6,6,6,6,6}\Big/5! = \frac{30!}{(6!)^5 \cdot 5!}$$

$$= 11{,}423{,}951{,}396{,}577{,}720.$$

c) Combinatorics: summary.

Each of the following things can be done in the given number of ways.

I) Permutations.

1) Arranging n things in order: $n!$

2) Arranging r things out of n in order:
$$\frac{n!}{(n-r)!} = n(n-1)(n-2)\cdots(n-r+1)$$

3) Arranging n things, in which n_1 are the same, n_2 are the same, ..., n_r are the same, in order:
$$\frac{n!}{n_1!\,n_2!\cdots n_r!} = \binom{n}{n_1, n_2, \ldots n_r}.$$

II) <u>Combinations.</u>

1) Choosing r things out of n:
$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

2) Placing n objects into distinct groups of sizes $n_1, n_2, \ldots, n_r$: again,
$$\binom{n}{n_1, n_2, \ldots, n_r}.$$