More combinatorics.

Part I: permutations, continued.

Example 1.

How many distinct arrangements are there of the letters in the word

sleeplessness??

(Treat all e's as the same, and all s's, and both l's.)

Solution

If all letters were different, the answer would be 13! But we're overcounting, since all e's are the same, etc.

Our overcount is by a factor of $4! \cdot 5! \cdot 2!$, since there are 4! permutations of the 4 e's, 5! of the 5 s's, and 2! of the 2 l's.

So the correct answer is

$$\frac{13!}{4! \cdot 5! \cdot 2!} = 1{,}081{,}080$$

arrangements.

In general:

> There are
>
> $$\frac{n!}{n_1! \cdot n_2! \cdots n_r!}$$
>
> permutations of n objects, where n_1 of them are the same, n_2 are the same, ... n_r are the same.

Combinatorics, part II: combinations.

Question:
in how many ways can we choose r objects (without considering order) from a collection of n objects (with $0 \le r \le n$)?

Answer:
if order mattered, the answer would be

$$n(n-1)(n-2)\cdots(n-r+1) = n!/(n-r)!,$$

as seen before. But order doesn't matter, so to compensate, we divide by the # of permutations of the r objects. So:

The number of ways of choosing r objects out of n is

$$\frac{n!}{r!(n-r)!}, \quad \text{denoted } \binom{n}{r}$$

"n choose r"

Examples.
2) Compute $\binom{7}{3}$ and $\binom{100}{3}$ without using a calculator until the last step.

3) a) How many different 5-card hands can be made from a standard, 52-card deck?
b) How many of these are full houses (3

cards of one value, two of another)?

c) How many are flushes (all of the same suit)?

Solutions

2) $\binom{7}{3} = \frac{7!}{3!\,4!} = \frac{7\cdot 6\cdot 5\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}{3!\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot \cancel{1}}$

$= \frac{7\cdot \cancel{6}\cdot 5}{\cancel{3}\cdot \cancel{2}\cdot 1} = 35.$ Also

97! cancels 97 of the factors in 100!

$\binom{100}{3} = \frac{100!}{3!\,97!} = \frac{100\cdot 99\cdot 98}{3!}$

$= \frac{100\cdot \overset{33}{\cancel{99}}\cdot \overset{49}{\cancel{98}}}{\cancel{3}\cdot \cancel{2}\cdot 1} = 100\cdot 33\cdot 49 = 161,700.$

3a) We choose 5 cards from 52; there are

$\binom{52}{5} = \frac{52!}{5!\,47!} = \frac{52\cdot 51\cdot \overset{10}{\cancel{50}}\cdot 49\cdot \overset{2}{\cancel{48}}}{\cancel{5}\cdot \cancel{4}\cdot \cancel{3}\cdot \cancel{2}\cdot 1}$

$= 52\cdot 51\cdot 10\cdot 49\cdot 2 = 2,598,960$

ways.

3b) $\underbrace{\binom{13}{1}}_{\text{choose one face value}}\cdot\underbrace{\binom{4}{3}}_{\text{choose 3 cards (out of 4) of that value}}\cdot\underbrace{\binom{12}{1}}_{\text{choose the remaining face value}}\cdot\underbrace{\binom{4}{2}}_{\text{choose two cards (out of 4) of that value}} = 13\cdot 4\cdot 12\cdot 6 = 3,744$ ways.

3c) Choose a suit, then 5 cards of that suit.
This gives $\binom{4}{1} \cdot \binom{13}{5} = 5148$ ways.