Prelude to probability:
Combinatorics (advanced counting).

(Idea: to see how likely something is, you might want to count the ways it can happen.)

Part I: permutations.

(a) A permutation of n objects is an ordered list, or arrangement, of these objects.

For example, we have 6 permutations of the letters X, Y, Z:

XYZ, XZY, YXZ, YZX, ZXY, ZYX.

And 24 permutations of the letters W, X, Y, Z:

WXYZ, WXZY, WYXZ, WYZX, WZXY, ... ZYXW.

DIY (Do It Yourself): complete this list!

In general, to count permutations of n objects, we:

pick the first object: n ways to do this
pick the second object: $n-1$ ways
pick the third object: $n-2$ ways
⋮
pick the next-to-last object: 2 ways
pick the last object: 1 way

These ways <u>multiply</u>*, so there are, in all,

$$n(n-1)(n-2)\cdots 2\cdot 1$$

denote this product by $n!$ ("n factorial")

permutations of n objects.

* The Basic Counting Principle (BCP) says: if there are n ways of doing Thing 1, and for each of these ways, there are m ways of doing Thing 2, then there are $n \cdot m$ ways of doing both Things together (or one followed by the other).

(The BCP extends to n Things, not just two.)

Example:

Suppose Dr. S. has 44 pair of sneakers, one for each day of class.

There are

$$44! = 44\cdot 43\cdot 42\cdot \ldots \cdot 3\cdot 2\cdot 1$$

$$\approx 2.66\times 10^{54}$$

ways he can wear a different pair each day.

(b) Now suppose we have n objects and we want to make a list of only r of them, where $0 \le r \le n$.

By the BCP, we can do this in

$$n(n-1)(n-2)\cdots(n-r+1)^{**}$$

ways (check: this product has exactly r factors).

Examples:

(i) The number of length-3 lists from the letters U, V, W, X, Y, Z is

$$6\cdot 5\cdot 4 = 120.$$

(ii) If Dr. S has 100 pair of sneakers, he can wear a different pair on each of 44 days in

$$100\cdot 99\cdot 98\cdots(100-44+1)$$
$$\approx 2.3\times 10^{81} \text{ ways.}$$

** Final note: the quantity

$$n(n-1)\cdots(n-r+1)$$

can also be written as $\dfrac{n!}{(n-r)!}$, since

$$\frac{n!}{(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)\cancel{(n-r)}\cancel{(n-r-1)}\cdots\cancel{2}\cdot\cancel{1}}{\cancel{(n-r)}\cancel{(n-r-1)}\cdots\cancel{2}\cdot\cancel{1}}$$

$$= n(n-1)(n-2)\cdots(n-r+1).$$

In other words:

(a) The number of permutations of n objects is $n!$.

(b) The number of length-r lists (also called r-permutations) made from n objects is

$$n!/(n-r)!.$$