Solutions to Selected Exercises, HW 8

Part A: Problems 4.54, 4.57, 4.60 from the text.

Exercise 4.54: The expected number of typographical errors on a page of a certain magazine is .2. What is the probability that the next page you read contains (a) 0 and (b) 2 or more typographical errors? Explain your reasoning.

Solution: The number X of errors on a page is $P(.2)$. So:

(a) $P(X=0) = \frac{.2^0}{0!}e^{-.2} = e^{-.2} = 0.8187.$

(b) $P(X \geq 2) = 1 - P(X=0) - P(X=1) = 1 - \frac{.2^0}{0!}e^{-.2} - \frac{.2^1}{1!}e^{-.2} = 0.0175.$

Exercise 4.57: State your assumptions. Suppose that the average number of cars abandoned weekly on a certain highway is 2.2. Approximate the probability that there will be:

(a) no abandoned cars in the next week;

(b) at least two abandoned cars in the next week.

Solution: The number Y of abandoned cars in a week is $P(2.2)$. So:

(a) $P(Y=0) = \frac{2.2^0}{0!}e^{-2.2} = e^{-2.2} = 0.1108.$

(b) $P(Y \geq 2) = 1 - P(Y=0) - P(Y=1) = 1 - \frac{2.2^0}{0!}e^{-2.2} - \frac{2.2^1}{1!}e^{-2.2} = 0.6454.$

Here, we are assuming that cars are abandoned independently of each other.

Exercise 4.60: Suppose that the number of accidents occurring on a highway each day is a Poisson random variable with parameter $\lambda = 3$.

(a) Find the probability that 3 or more accidents occur today.

(b) Repeat part (a) under the assumption that at least 1 accident occurs today.

Solution: The number Z of accidents in a day is $P(3)$. So:

(a) $P(Z \geq 3) = 1 - P(Z=0) - P(Z=1) - P(Z=2) = 1 - \frac{3^0}{0!}e^{-3} - \frac{3^1}{1!}e^{-3} - \frac{3^2}{2!}e^{-3} = 0.5768.$

(b)

$$P(Z \geq 3 | Z \geq 1) = \frac{P(Z \geq 3 \text{ and } Z \geq 1)}{P(Z \geq 1)} = \frac{0.5768}{1 - P(Z=0)} = \frac{0.5768}{1 - \frac{3^0}{0!}e^{-3}} = 0.607032.$$

Part B.

Recall the following. Suppose a certain event happens, on average, λ times per interval of a given extent. Let X be the number of times that the event *actually* happens in such an interval. Then we say "X is Poisson of parameter λ," or simply "X is $P(\lambda)$," and we have the formula

$$P(X = k) = \frac{\lambda^k}{k!} e^{-\lambda} \qquad (k = 0, 1, 2, 3, \ldots). \tag{$*$}$$

For the first four problems we refer to a 1911 experiment of Ernest Rutherford et al., who determined that, on average, a certain sample of polonium emitted 3.8715 α-rays every 7.5 seconds.

1. Find the probability that, in a 7.5-second interval, this sample emits 6 α-rays.

   We have $\lambda = 3.8715$, so

   $$P(X = 6) = \frac{3.8715^6}{6!} e^{-3.8715} \approx 0.0974 = 9.74\%.$$

2. Fill in the blanks: Note that $4 \times 7.5 =$ ____30____. So, since an average of 3.8715 α-rays are emitted every 7.5 seconds, we would expect that, on average, $4 \times 3.8715 =$ ____15.486____ α-rays will be emitted every 30 seconds.

3. Let Y be the number of α-rays emitted in a 30 second interval. Using your answer to question 2 above, find $P(Y = 24)$. Hint: your parameter λ here is computed in question 2 above.

   We have $\lambda = 15.486$, so

   $$P(Y = 24) = \frac{15.486^{24}}{24!} e^{-15.486} \approx 0.0110 = 1.1\%.$$

4. True or false: the probability of getting 24 α-rays in 30 seconds is the same as the probability of getting 6 α-rays in 7.5 seconds. Explain how you know. Is this result surprising to you? If so, why? If not, why?

   False. By the previous two problems, the probability of getting 24 α-rays in 30 seconds is about 1.1%, while the probability of getting 6 α-rays in 7.5 seconds is about 9.74%. This might seem surprising: shouldn't the probability of getting a certain number in a certain amount of time be the same as the probability as getting four times as many in four times the amount of time? Well no: getting 6 in 7.5 seconds is relatively rare, so having that happen four times in a row is even more rare.

5. Suppose X is a Poisson random variable of parameter λ, where λ is a positive integer. Find an integer k such that $P(X = k) = P(X = k+1)$. (Your answer should be in terms of the parameter λ.)

   Find k. Hint: use formula **(∗)**, from the first page of this problem set, to write down formulas for $P(X = k)$ and $P(X = k + 1)$. Set them equal and solve for k.

   Setting $P(X = k) = P(X = k + 1)$ gives

$$\frac{\lambda^k}{k!} e^{-\lambda} = \frac{\lambda^{k+1}}{(k+1)!} e^{-\lambda}.$$

   Divide both sides by $e^{-\lambda}$ and by λ^k to get

$$\frac{1}{k!} = \frac{\lambda}{(k+1)!}$$

   Multiply both sides by $(k + 1)!$, and note that $(k + 1)!/k! = k + 1$, to get $k + 1 = \lambda$, or $k = \lambda - 1$. So $P(X = \lambda - 1) = P(X = \lambda)$.

In discussing the Poisson distribution in class on Monday, we first came up with the formula

$$P(X = k) = \lim_{N\to\infty} \binom{N}{k} \left(\frac{\lambda}{N}\right)^k \left(1 - \frac{\lambda}{N}\right)^{N-k} \qquad (**)$$

when X is $P(\lambda)$, and then claimed that **(∗∗)** yields **(∗)**. In class on Wednesday, we directly showed that this is true directly for some small values of k. The purpose of the next few exercises is to show that this holds for $k = 4$.

6. Show that

$$\binom{N}{4} = \frac{N(N-1)(N-2)(N-3)}{4!}.$$

   Hint: start with the definition of $\binom{N}{4}$.

   Note that

$$\frac{N!}{(N-4)!} = \frac{N(N-1)(N-2)(N-3)(N-4)(N-5)\cdots 2\cdot 1}{(N-4)(N-5)\cdots 2\cdot 1}$$
$$= N(N-1)(N-2)(N-2)(N-3).$$

   So

$$\binom{N}{4} = \frac{N!}{4!(N-4)!} = \frac{N(N-1)(N-2)(N-4)}{4!}.$$

7. Use your answer to question 6 above, together with **(∗∗)** (at the top of the previous page), to show that, if X is $P(\lambda)$, then

$$P(X=4) = \frac{\lambda^4}{4!} \cdot \lim_{N\to\infty} \frac{N(N-1)(N-2)(N-3)}{N^4} \cdot \left(1 - \frac{\lambda}{N}\right)^{N-4}.$$

By **(∗∗)** and by question 6,

$$\begin{aligned} P(X=4) &= \lim_{N\to\infty} \binom{N}{4} \left(\frac{\lambda}{N}\right)^4 \left(1-\frac{\lambda}{N}\right)^{N-4} \\ &= \lim_{N\to\infty} \frac{N(N-1)(N-2)(N-3)}{4!} \left(\frac{\lambda}{N}\right)^4 \left(1-\frac{\lambda}{N}\right)^{N-4}. \end{aligned}$$

Everything that doesn't involve N – namely, the λ^4 in the numerator and the 4! in the denominator – can be pulled outside of the limit, so we get

$$P(X=4) = \frac{\lambda^4}{4!} \lim_{N\to\infty} \frac{N(N-1)(N-2)(N-3)}{N^3} \left(1-\frac{\lambda}{N}\right)^{N-4}.$$

8. What is

$$\lim_{N\to\infty} \frac{N(N-1)(N-2)(N-3)}{N^4}\ ?$$

Explain. (One way to do this is to multiply out the numerator, and then use l'Hôpital's Rule, treating N as a real variable. If you have another way of doing it, feel free, but please explain your method.)

As suggested, we multiply out the numerator, and then apply l'Hôpital's Rule repeatedly, like this:

$$\begin{aligned} &\lim_{N\to\infty} \frac{N(N-1)(N-2)(N-3)}{N^4} = \lim_{N\to\infty} \frac{N^4 - 6N^3 + 11N^2 - 6N}{N^4} \\ &= \lim_{N\to\infty} \frac{12N^2 - 36N + 22}{4N^3} = \lim_{N\to\infty} \frac{24N - 36}{24N} \\ &= \lim_{N\to\infty} \frac{24}{24} = 1. \end{aligned}$$

9. What is

$$\lim_{N\to\infty} \left(1-\frac{\lambda}{N}\right)^{N-4}\ ?$$

Hint: You may use the facts, from calculus, that if x and a are *constants* (with respect to N), then

$$\lim_{N\to\infty} \left(1+\frac{x}{N}\right)^N = e^x, \quad \lim_{N\to\infty} \left(1+\frac{x}{N}\right)^a = 1.$$

By the hint,

$$\begin{aligned}\lim_{N\to\infty}\left(1-\frac{\lambda}{N}\right)^{N-4} &= \left(\lim_{N\to\infty}\left(1-\frac{\lambda}{N}\right)^{N}\right)\left(\lim_{N\to\infty}\left(1-\frac{\lambda}{N}\right)^{-4}\right)\\ &= e^{-\lambda}\cdot 1 = e^{-\lambda}.\end{aligned}$$

10. Put your answers to question 8 and 9 above into your answer from question 7 above to find a simple formula (with no limits in it) for $P(X=4)$. Hint: you know what your answer should be, by **(∗)**.

By problems 8 and 9, and the fact that the limit of a product equals the product of the limits,

$$\begin{aligned}P(X=4) &= \frac{\lambda^4}{4!}\cdot\left(\lim_{N\to\infty}\frac{N(N-1)(N-2)(N-3)}{N^4}\right)\cdot\left(\lim_{N\to\infty}\left(1-\frac{\lambda}{N}\right)^{N-4}\right)\\ &= \frac{\lambda^4}{4!}\cdot 1\cdot e^{-\lambda} = \frac{\lambda^4}{4!}e^{-\lambda}.\end{aligned}$$