Homework Assignment #8B: The Poisson Distribution

Please complete and turn in on Wednesday, October 30. You may write your answers on these pages or on your own paper.

Recall the following. Suppose a certain event happens, on average, λ times per interval of a given extent. Let X be the number of times that the event *actually* happens in such an interval. Then we say "X is Poisson of parameter λ," or simply "X is $P(\lambda)$," and we have the formula

$$P(X = k) = \frac{\lambda^k}{k!}\, e^{-\lambda} \qquad (k = 0, 1, 2, 3, \ldots). \qquad (*)$$

For the first four problems we refer to a 1911 experiment of Ernest Rutherford et al., who determined that, on average, a certain sample of polonium emitted 3.8715 α-rays every 7.5 seconds.

1. Find the probability that, in a 7.5-second interval, this sample emits 6 α-rays.

2. Fill in the blanks: Note that $4 \times 7.5 =$ ______________. So, since an average of 3.8715 α-rays are emitted every 7.5 seconds, we would expect that, on average, $4 \times 3.8715 =$ __________________ α-rays will be emitted every 30 seconds.

3. Let Y be the number of α-rays emitted in a 30 second interval. Using your answer to question 2 above, find $P(Y = 24)$. Hint: your parameter λ here is computed in question 2 above.

4. True or false: the probability of getting 24 α-rays in 30 seconds is the same as the probability of getting 6 α-rays in 7.5 seconds. Explain how you know. Is this result surprising to you? If so, why? If not, why?

5. Suppose X is a Poisson random variable of parameter λ, where λ is a positive integer. Find an integer k such that $P(X = k) = P(X = k+1)$. (Your answer should be in terms of the parameter λ.)

   Hint: use formula **(∗)**, from the first page of this problem set, to write down formulas for $P(X = k)$ and $P(X = k + 1)$. Set them equal and solve for k.

In discussing the Poisson distribution, we first came up with the formula

$$P(X = k) = \lim_{N\to\infty} \binom{N}{k} \left(\frac{\lambda}{N}\right)^k \left(1 - \frac{\lambda}{N}\right)^{N-k} \tag{$**$}$$

when X is $P(\lambda)$, and then claimed that $(**)$ yields $(*)$. We later showed directly that this is true for some small values of k. The purpose of the next few exercises is to show that this holds for $k = 4$.

6. Show that

$$\binom{N}{4} = \frac{N(N-1)(N-2)(N-3)}{4!}.$$

Hint: start with the definition of $\binom{N}{4}$.

7. Use your answer to question 6 above, together with **(∗∗)** (at the top of the previous page), to show that, if X is $P(\lambda)$, then

$$P(X=4) = \frac{\lambda^4}{4!} \cdot \lim_{N\to\infty} \frac{N(N-1)(N-2)(N-3)}{N^4} \cdot \left(1 - \frac{\lambda}{N}\right)^{N-4}.$$

8. What is

$$\lim_{N\to\infty} \frac{N(N-1)(N-2)(N-3)}{N^4} \ ?$$

Explain. (One way to do this is to multiply out the numerator, and then use l'Hôpital's Rule, treating N as a real variable. If you have another way of doing it, feel free, but please explain your method.)

9. What is

$$\lim_{N\to\infty}\left(1-\frac{\lambda}{N}\right)^{N-4} \;?$$

Hint: You may use the facts, from calculus, that if x and a are *constants* (with respect to N), then

$$\lim_{N\to\infty}\left(1+\frac{x}{N}\right)^{N}=e^{x}, \quad \lim_{N\to\infty}\left(1+\frac{x}{N}\right)^{a}=1.$$

10. Put your answers to question 8 and 9 above into your answer from question 7 above to find a simple formula (with no limits in it) for $P(X = 4)$. Hint: you know what your answer should be, by (∗).