Solutions to Selected Exercises, HW #7

Assignment.

Chapter 4 problems, starting on p. 175:

Problems 41, 43, 44, 46, 49, 50.

Note: We will use the shorthand "X is $B(n,p)$" to mean X is a binomial random variable with n trials and, in each trial, $P(\text{success}) = p$. (To express the same thing, the book says "X is a binomial random variable with parameters (n,p).")

Exercise 41: On a multiple-choice exam with 3 possible answers for each of the 5 questions, what is the probability that a student will get 4 or more correct answers just by guessing?

Solution: As noted in the hints, if X is the number correct, then X is $B(5,1/3)$. So

$$P(X \geq 4) = P(X = 4) + P(X = 5)$$
$$= \binom{5}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^1 + \binom{5}{5}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^0 = 0.0453.$$

Exercise 43: A and B will take the same 10-question examination. Each question will be answered correctly by A with probability .7, independently of her results on other questions. Each question will be answered correctly by B with probability .4, independently both of her results on the other questions and on the performance of A.

(a) Find the expected number of questions that are answered correctly by both A and B.

(b) Find the variance of the number of questions that are answered correctly by either A or B.

Solution: **(a)** As noted in the hints, the number Z that both get correct is $B(10,0.28)$. So

$$E[Z] = np = 10 \cdot 0.28 = 2.8.$$

(b) As noted in the hints,

$$P(A \text{ or } B \text{ is correct}) = P(A \text{ is correct}) + P(B \text{ is correct}) - P(A \text{ and } B \text{ are correct})$$
$$= .7 + .4 = .28 = .82.$$

So, if Q is the number correct by either A or B, then

$$\text{Var}[Q] = np(1-p) = 10 \cdot 0.82 \cdot (1 - 0.82) = 1.476.$$

Exercise 44: A communications channel transmits the digits 0 and 1. However, due to static, the digit transmitted is incorrectly received with probability .2. Suppose that we want to transmit an important message consisting of one binary digit. To reduce the chance of error, we transmit 00000 instead of 0 and 11111 instead of 1. If the receiver of the message uses "majority" decoding, what is the probability that the message will be wrong when decoded? What independence assumptions are you making?

Solution: As noted in the hints, the number X of digits received incorrectly is $B(5,.2)$. The message will be received incorrectly if $X > 2$. We have

$$\begin{aligned} P(X > 2) &= P(X = 3) + P(X = 4) + P(X = 5) \\ &= \binom{5}{3} \cdot .2^3 \cdot .8^2 + \binom{5}{4} \cdot .2^4 \cdot .8^1 + \binom{5}{5} \cdot .2^5 \cdot .8^0 = 0.0579. \end{aligned}$$

Here, we're assuming independence of errors in each digit.

Exercise 46: A student is getting ready to take an important oral examination and is concerned about the possibility of having an "on" day or an "off" day. He figures that if he has an on day, then each of his examiners will pass him, independently of one another, with probability .8, whereas if he has an off day, this probability will be reduced to .4. Suppose that the student will pass the examination if a majority of the examiners pass him. If the student believes that he is twice as likely to have an off day as he is to have an on day, should he request an examination with 3 examiners or with 5 examiners?

Solution: Let X be the number of examiners, out of 3, who pass the student on an on day, and Y the number of examiners who pass the student on an off day. Then X is $B(3,.8)$ and Y is $B(3,.4)$. So as noted in the hints,

$$\begin{aligned} &P(\text{pass with 3 examiners}) \\ &= P(\text{on day})P(\text{pass with 3 examiners given on day}) \\ &+ P(\text{off day})P(\text{pass with 3 examiners given off day}) \\ &= \frac{1}{3} \cdot P(X \geq 2) + \frac{2}{3} \cdot P(Y \geq 2) \\ &= \frac{1}{3} \cdot \big(P(X = 2) + P(X = 3)\big) + \frac{2}{3} \cdot \big(P(Y = 2) + P(Y = 3)\big) \\ &= \frac{1}{3} \cdot \left(\binom{3}{2} \cdot .8^2 \cdot .2^1 + \binom{3}{3} \cdot .8^3 \cdot .2^0\right) + \frac{2}{3} \cdot \left(\binom{3}{2} \cdot .4^2 \cdot .6^1 + \binom{3}{3} \cdot .4^3 \cdot .6^0\right) \\ &= 0.5333. \end{aligned}$$

Similarly,

$$\begin{aligned}
&P(\text{pass with 5 examiners})\\
&= P(\text{on day})P(\text{pass with 5 examiners given on day})\\
&+ P(\text{off day})P(\text{pass with 5 examiners given off day})\\
&= \frac{1}{3}\cdot P(X\geq 3) + \frac{2}{3}\cdot P(Y\geq 3)\\
&= \frac{1}{3}\cdot \big(P(X=3)+P(X=4)+P(X=5)\big) + \frac{2}{3}\cdot \big(P(Y=3)+P(Y=4)+P(Y=5)\big)\\
&= \frac{1}{3}\cdot \left(\binom{5}{3}\cdot .8^3\cdot .2^2 + \binom{5}{4}\cdot .8^4\cdot .2^1 + \binom{5}{5}\cdot .8^5\cdot .2^0\right)\\
&+ \frac{2}{3}\cdot \left(\binom{5}{3}\cdot .4^3\cdot .6^2 + \binom{5}{4}\cdot .4^4\cdot .6^1 + \binom{5}{5}\cdot .4^5\cdot .6^0\right)\\
&= 0.5257.
\end{aligned}$$

The student is marginally better off with 3 examiners.

Exercise 49: It is known that diskettes produced by a certain company will be defective with probability .01, independently of one another. The company sells the diskettes in packages of size 10 and offers a money-back guarantee that at most 1 of the 10 diskettes in the package will be defective. The guarantee is that the customer can return the entire package of diskettes if he or she finds more than 1 defective diskette in it. If someone buys 3 packages, what is the probability that he or she will return exactly 1 of them?

Solution: If X is the number defective in a *single* package, then X is $B(10,.01)$. The probability that a package is returned is then

$$\begin{aligned}
P(X>1) = 1 - P(X=0) - P(X=1) &= 1 - \binom{10}{0}\cdot .01^0\cdot .99^{10} - \binom{10}{1}\cdot .01^1\cdot .99^{10}\\
&= 0.0042662.
\end{aligned}$$

Now let Y be the number of packages returned if 3 are bought. Then Y is $B(3,0.0042662)$. So

$$P(Y=1) = \binom{3}{1}\cdot 0.0042662^1\cdot (1-0.0042662)^2 = 0.0126896.$$

Exercise 50: When coin 1 is flipped, it lands on heads with probability .4; when coin 2 is flipped, it lands on heads with probability .7. One of these coins is randomly chosen and flipped 10 times.

(a) What is the probability that the coin lands on heads on exactly 7 of the 10 flips?

(b) Given that the first of these 10 flips lands heads, what is the conditional probability that exactly 7 of the 10 flips land on heads?

Solution: (a) If X is the number of heads using Coin 1, then X is $B(10, .4)$. If Y is the number of heads using Coin 2, then Y is $B(10, .7)$. So

$$\begin{aligned} P(7 \text{ heads}) &= P(\text{Coin } 1)P(7 \text{ heads given Coin } 1) \\ &\quad + P(\text{Coin } 2)P(7 \text{ heads given Coin } 2) \\ &= \frac{1}{2} \cdot \binom{10}{7} \cdot .4^7 \cdot .6^3 + \frac{1}{2} \cdot \binom{10}{7} \cdot .7^7 \cdot .3^3 = 0.154648. \end{aligned}$$

Solution: (b) As noted in the hints,

$$P(7 \text{ heads} \mid \text{first is heads}) = \frac{P(7 \text{ heads and first is heads})}{P(\text{first is heads})} = \frac{P(7 \text{ heads and first is heads})}{.55}.$$

Further,

$$\begin{aligned} &P(7 \text{ heads and first is heads}) \\ &= P(\text{Coin } 1)P(7 \text{ heads and first is heads given Coin } 1) \\ &+ P(\text{Coin } 2)P(7 \text{ heads and first is heads given Coin } 2) \\ &= \frac{1}{2} \cdot P(7 \text{ heads and first is heads given Coin } 1) \\ &+ \frac{1}{2} \cdot P(7 \text{ heads and first is heads given Coin } 2). \end{aligned}$$

But

$$\begin{aligned} &P(7 \text{ heads and first is heads given Coin } 1) \\ &= P(\text{first is heads for Coin } 1) \cdot P(6 \text{ of the remaining } 9 \text{ are heads for Coin } 1) \\ &= .4 \cdot \binom{9}{6} .4^6 .6^3 = 0.0297. \end{aligned}$$

Similarly,

$$\begin{aligned} &P(7 \text{ heads and first is heads given Coin } 2) \\ &= P(\text{first is heads for Coin } 2) \cdot P(6 \text{ of the remaining } 9 \text{ are heads for Coin } 2) \\ &= .7 \cdot \binom{9}{6} .7^6 .3^3 = 0.1868. \end{aligned}$$

So

$$\begin{aligned} &P(7 \text{ heads and first is heads}) \\ &= \frac{1}{2} \cdot 0.0297 + \frac{1}{2} \cdot 0.1868 = 0.1083. \end{aligned}$$

Then, finally,

$$P(7 \text{ heads} \mid \text{first is heads}) = \frac{P(7 \text{ heads and first is heads})}{.55} = \frac{0.1083}{.55} = 0.1968.$$