Solutions to Selected Exercises, HW #6

Assignment.

A. Chapter 4 problems, starting on p. 175:

Problems 4.4, 4.22, 4.35b, 4.37, 4.38, 4.39.

B. Additional problems.

Part A.

Exercise 4: Five men and 5 women are ranked according to their scores on an examination. Assume that no two scores are alike and all 10! possible rankings are equally likely. Let X denote the highest ranking achieved by a woman. (For instance, $X = 1$ if the top-ranked person is female.) Find $P(X = i)$, $i = 1, 2, 3, .., 8, 9, 10$.

Solution: For example, $P(X = 1) = 0.5$, since there's an equal chance that the highest ranked is a man or a woman.

Arguing as in the hints given, we have

$$\begin{aligned}
P(X = 2) &= \frac{5 \cdot 5 \cdot 8!}{10!} = \frac{5 \cdot 5}{10 \cdot 9} = \frac{5}{18} = 0.2778, \\
P(X = 3) &= \frac{5 \cdot 4 \cdot 5 \cdot 7!}{10!} = \frac{5 \cdot 4 \cdot 5}{10 \cdot 9 \cdot 8} = \frac{5}{36} = 0.1489, \\
P(X = 4) &= \frac{5 \cdot 4 \cdot 3 \cdot 5 \cdot 6!}{10!} = \frac{5 \cdot 4 \cdot 3 \cdot 5}{10 \cdot 9 \cdot 8 \cdot 7} = \frac{5}{84} = 0.0595, \\
P(X = 5) &= \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 5 \cdot 5!}{10!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 5}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6} = \frac{5}{252} = 0.0198, \\
P(X = 6) &= \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 5 \cdot 4!}{10!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 5}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5} = \frac{1}{252} = 0.0040, \\
P(X = 7) &= P(X = 8) = P(X = 9) = P(X = 10) = 0.
\end{aligned}$$

(There are only 5 men, so the highest ranking achieved by a woman is at least 6.)

Exercise 22: 4.22. Suppose that two teams play a series of games that ends when one of them has won i games. Suppose that each game played is, independently, won by team A with probability p. Find the expected number of games that are played when **(a)** $i = 2$. Show that this number is maximized when $p = 1/2$.

Solution: As per the hints, we have

$$P(X = 2) = p \cdot p + (1 - p) \cdot (1 - p) = p^2 + (1 - p)^2.$$

Also, as explained in the hints, X could be 3, if the teams split the first two games and one team or another wins the third. That is, the sequence of wins could be ABA or BAB or ABB or BAA (Again, B is the other team.) Using this to compute $P(X = 3)$, we find that

$$\begin{aligned}
P(X = 3) &= p \cdot (1 - p) \cdot p + (1 - p) \cdot p \cdot (1 - p) \\
&\quad + p \cdot (1 - p)^2 + (1 - p) \cdot p^2 = 2p(1 - p)^2 + 2p^2(1 - p).
\end{aligned}$$

So

$$E[X] = 2 \cdot (p^2 + (1-p)^2) + 3 \cdot (2p(1-p)^2 + 2p^2(1-p)) = 2 + 2p - 2p^2.$$

To maximize $E[X]$ with respect to p, call it $f(p)$: $f(p) = 2 + 2p - 2p^2$. Then $f'(p) = 2 - 4p$. Setting this equal to zero and solving gives us the critical point $p = 1/2$. Since $f''(p) = -4 < 0$,we know that this critical point is a local maximum, and therefore, since it's the only critical point, it must be the global maximum on the interval $0 \leq p \leq 1$.

Incidentally, we have $E[X] = f(1/2) = 2 + 2 \cdot \frac{1}{2} = 2 \cdot \left(\frac{1}{2}\right)^2 = 2.5$. That is: at most, we would expect it to take 2.5 games for the series to complete.

Exercise 35b: A box contains 5 red and 5 blue marbles. Two marbles are withdrawn randomly. If they are the same color, then you win \$1.10; if they are different colors, then you win \$−1.00. (That is, you lose \$1.00.) Calculate the variance of the amount you win.

Solution: In the previous assignment (Homework #5, Exercise 35a), you computed that

$$P(X = \$1.10) = \frac{5}{10} \cdot \frac{4}{9} + \frac{5}{10} \cdot \frac{4}{9} = \frac{4}{9}; \qquad P(X = -\$1) = \frac{5}{10} \cdot \frac{5}{9} + \frac{5}{10} \cdot \frac{5}{9} = \frac{5}{9},$$

and

$$\mu = E[X] = -\frac{1}{15}.$$

So by the definition of variance,

$$\mathrm{Var[X]} = \sum_{\substack{\text{values} \\ x \text{ of } X}} (x-\mu)^2 P(X = x) = \left(1.10 + \tfrac{1}{15}\right)^2 \cdot \frac{4}{9} + \left(-1 + \tfrac{1}{15}\right)^2 \cdot \frac{5}{9} = 1.0889$$

(in dollars).

Exercise 37: Consider Problem 4.22 with $i = 2$. Find the variance of the number of games played, and show that this number is maximized when $p = 1/2$.

Solution: In Exercise 4.22 above, we showed that

$$P(X = 2) = p^2 + (1-p)^2; \quad P(X = 3) = 2p(1-p)^2 + 2p^2(1-p); \quad E[X] = 2 + 2p - 2p^2.$$

So

$$\begin{aligned}\mathrm{Var[X]} &= \left(2 - (2 + 2p - 2p^2)\right)^2 \cdot (p^2 + (1-p)^2) + \left(3 - (2 + 2p - 2p^2)\right)^2 \cdot (2p(1-p)^2 + 2p^2(1-p)) \\ &= 2p - 6p^2 + 8p^3 - 4p^4.\end{aligned}$$

Then

$$f'(p) = 2 - 12p + 24p^2 - 16p^3.$$

It's probably not obvious, but this factors into $-2(-1+2p)^3$. Clearly this quantity equals zero when $p = 1/2$. Now $f''(p) = -12 + 48p - 48p^2 = -12(1-2p)^2$, so $f''(1/2) = 0$, so we can't use the second derivative test here. However, we can compare $f(p)$ at $p = 1/2$ to $f(p)$ at the endpoints $p = 0$ and $p = 1$ of the domain for p, and we find $f(0) = f(1) = 0$ while $f(1/2) = 1/4$, so clearly the maximum of $f(p)$ occurs at $p = 1/2$.

Exercise 38: Find $\mathrm{Var}[X]$ and $\mathrm{Var}[Y]$ for X and Y as given in Problem 4.21.

Solution: From Homework #5, Exercise 4.21,

$$P(X=40) = \frac{40}{148}, \quad P(X=33) = \frac{33}{148}, \quad P(X=25) = \frac{25}{148}, \quad P(X=50) = \frac{50}{148},$$
$$E[X] = 39.2838,$$
$$P(Y=40) = P(Y=33) = P(Y=25) = P(Y=50) = \frac{1}{4}, \quad E[Y] = 37.$$

So

$$\begin{aligned}\mathrm{Var}[X] &= (40-39.2838)^2 \cdot \frac{40}{148} + (33-39.2838)^2 \cdot \frac{33}{148} + (25-39.2838)^2 \cdot \frac{25}{148} \\ &\quad + (50-39.2838)^2 \cdot \frac{50}{148} = 82.2033, \\ \mathrm{Var}[Y] &= (40-37)^2 \cdot \frac{1}{4} + (33-37)^2 \cdot \frac{1}{4} + (25-37)^2 \cdot \frac{1}{4} + (50-37)^2 \cdot \frac{1}{4} = 84.5000.\end{aligned}$$

Exercise 39: If $E[X] = 1$ and $\mathrm{Var}[X] = 5$, find **(a)** $E[(2+X)^2]$; **(b)** $\mathrm{Var}[4+3X]$.

By the hint,
$$E[(2+X)^2] = E[4+4X] + E[X^2].$$

By Corollary 4.1 on page 132,

$$E[4+4X] = 4 + 4E[X] = 4 + 4 \cdot 1 = 8.$$

Also, by the boxed formula on page 132:

$$E[X^2] = \mathrm{Var}[X] + (E[X])^2 = 5 + 1^2 = 6.$$

So
$$E[(2+X)^2] = 8 + 6 = 14.$$

(b) By the formula near the bottom of page 136,

$$\mathrm{Var}[4+3X] = 3^2 \cdot \mathrm{Var}[X] = 9 \cdot 5 = 45.$$

Part B.

Exercise 1: What is the expected value of the number of times that two adjacent letters will be the same in a random permutation of the eleven letters of the word Mississippi?

Solution: For $i = 1, 2, 3, \ldots, 10$, let

$$X_i = \begin{cases} 1 & \text{if the } i\text{th and the } (i+1)\text{st letters are the same,} \\ 0 & \text{if not.} \end{cases}$$

Let $X = X_1 + X_2 + \cdots + X_{10}$: then X is the number of times two adjacent letters will be the same.

As per the hints,

$$P(X_1 = 1) = \frac{6+6+1}{55} = \frac{13}{55}.$$

So

$$E(X_1) = 1 \cdot \frac{13}{55} + 0 \cdot P(X_1 = 0) = \frac{13}{55} \approx 0.2364.$$

Similarly, $E(X_i) = 0.2364$ for each i. So

$$E(X) = 10 \cdot 0.2364 = 2.364.$$

Exercise 4: What is the expected number of times that two consecutive numbers will show up in a Lotto drawing of six different numbers from the numbers 1,2,$\cdots$,45?

Solution: For $i = 1, 2, 3, \ldots 44$, let

$$Y_i = \begin{cases} 1 & \text{if the numbers } i \text{ and } i+1 \text{ both show up,} \\ 0 & \text{if not.} \end{cases}$$

Let $Y = Y_1 + Y_2 + \cdots + Y_{44}$: then Y is the number of times two consecutive numbers will show up.

Now what is $E(Y_1)$? This is the expected number of times that both 1 and 2 show up. There is 1 way in which a 1 can show up, and 1 way in which a 2 can show up, and assuming a 1 and a 2 show up, there are $\binom{43}{4}$ ways of choosing the remaining 4 numbers. Also, there are $\binom{45}{6}$ possible ways of selecting 6 numbers from the 45. So

$$P(Y_1 = 1) = \frac{1 \cdot 1 \cdot \binom{43}{4}}{\binom{45}{6}}.$$

So

$$E(Y_1) = 1 \cdot P(Y_1 = 1) + 0 = \frac{1 \cdot 1 \cdot \binom{43}{4}}{\binom{45}{6}}.$$

Similarly,

$$E(Y_i) = \frac{1 \cdot 1 \cdot \binom{43}{4}}{\binom{45}{6}}$$

for any i. So

$$E(Y) = 44 \cdot \frac{1 \cdot 1 \cdot \binom{43}{4}}{\binom{45}{6}} = \frac{2}{3} \approx 0.6667.$$