Hints for Selected Exercises, HW #6

Assignment.

A. Chapter 4 problems, starting on p. 175:

Problems 4.4, 4.22, 4.35b, 4.37, 4.38, 4.39.

B. Additional problems.

Part A.

Exercise 4: For example, $P(X = 1) = 0.5$, since there's an equal chance that the highest ranked is a man or a woman.

To compute $P(X = 2)$, you could argue as follows. There are 10! possible rankings, and the number of these rankings where a woman ranks second highest is $5 \cdot 5 \cdot 8!$. (If a woman ranks second highest, then the highest ranking goes to a man, and there are 5 men to choose from; the next highest ranking goes to a woman, and there are 5 women to choose from. The remaining 8 can be ranked in any order.) So

$$P(X = 2) = \frac{5 \cdot 5 \cdot 8!}{10!} = \frac{5 \cdot 5}{10 \cdot 9} = \frac{5}{18} = 0.2778.$$

Similarly, to have $X = 3$, the highest two ranked must be men, and the third ranked must be a woman. There are 5 choices for the first man, leaving 4 for the next man, and then 5 for the woman, and then 7! for the remaining 7. So

$$P(X = 3) = \frac{5 \cdot 4 \cdot 5 \cdot 7!}{10!} = \frac{5 \cdot 4 \cdot 5}{10 \cdot 9 \cdot 8} = \frac{5}{36} = 0.1389.$$

And so on.

Exercise 22: Consider the case $i = 2$. We are saying that it takes 2 *wins* for the series to end. Let X be the number of *games* it takes for the series to end. X can't be 1, since neither team will have won two games after just one game. X could be 2, if the same team wins both of the first two games. What is the probability of this? Well, team A might win both, or the other team, team B, might win both. Team A has probability p of winning each one, so team B has probability $1 - p$ of winning each one. So

$$P(X = 2) = p \cdot p + (1 - p) \cdot (1 - p) = p^2 + (1 - p)^2.$$

Next, X could be 3, if the teams split the first two games and one team or another wins the third. That is, the sequence of wins could be ABA or BAB or ABB or BAA. Use this to compute $P(X = 3)$, much as we calculated $P(X = 2)$ above. From all of this information you can then calculate $E[X]$. If you multiply everything out and simplify, you should get $2 + 2p - 2p^2$.

The problem asks you to maximize $E[X]$. Do it with calculus! Let $f(p) = 2+2p-2p^2$. Set $f'(p) = 0$ to find where the maximum value of this function occurs. (Use the second derivative test to show you have a maximum instead of a minimum.)

Exercise 35b: If X is the amount won in dollars, then you computed the pmf for X and the expected value of X in part a of this exercise, in your previous assignment. See the solutions to HW #5. These numbers should help you compute $\text{Var}[X]$.

Exercise 37: Most of the information you need comes from Exercise 22 above.

Exercise 38: You'll need to know $E[X]$ and $E[Y]$, which you computed in your previous assignment. See the solutions to Exercise 21, HW #5.

Exercise 39(a):

$$E[(2+X)^2] = E[2+4X+X^2] = E[2+4X] + E[X^2],$$

by the sum rule. You can compute $E[2+4X]$ using what you're given about $E[X]$, together with Corollary 4.1 on page 132. You can compute $E[X^2]$ using what you're given about $E[X]$ and $\text{Var}[X]$, together with the boxed formula on page 132:

$$\text{Var}[X] = E[X^2] - (E[X])^2.$$

Part B.

Exercise 1: For $i = 1, 2, 3, \ldots, 10$, let

$$X_i = \begin{cases} 1 & \text{if the } i\text{th and the } (i+1)\text{st letters are the same,} \\ 0 & \text{if not.} \end{cases}$$

Let $X = X_1 + X_2 + \cdots + X_{10}$: then X is the number of times two adjacent letters will be the same.

What's the probability that $X_1 = 1$? This is the probability that the first two letters are either both i's, both s's, or both p's. (They can't both be m's, since there's only one m.) There are $\binom{4}{2} = 6$ ways for them both to be i's (since there are 4 i's), $\binom{4}{2} = 6$ ways for them both to be s's (since there are 4 s's), and $\binom{2}{2} = 1$ way for them both to be p's (since there are 2 p's). There are $\binom{11}{2} = 55$ total ways to choose 2 letters from the 11, so

$$P(X_1 = 1) = \frac{6+6+1}{55} = \frac{13}{55}.$$

You should be able to compute $E[X_1]$ from this. (If you know $P(X_1 = 1)$, then you know $P(X_1 = 0)$. On the other hand, you don't even need to know $P(X_1 = 0)$, since the term that comes up in the definition of $E[X]$ is $0 \cdot P(X_1 = 0)$, which is 0 regardless.) $E[X_i]$ is the same for all values of i in question.

Exercise 2: For $i = 1, 2, 3, \ldots 44$, let

$$Y_i = \begin{cases} 1 & \text{if the numbers } i \text{ and } i+1 \text{ both show up,} \\ 0 & \text{if not.} \end{cases}$$

Let $Y = Y_1 + Y_2 + \cdots + Y_{44}$: then Y is the number of times two consecutive numbers will show up.

What is $P(Y_1 = 1)$? Well, there's only 1 way of choosing a 1, and one way of choosing a 2. Once these are chosen, how many ways are there of choosing the remaining 4 numbers? And how many ways are there of choosing 6 numbers from 45 overall? From this information you can compute $P(Y_1 = 1)$. You can then compute $E[Y_1]$ from this. $E[Y_i]$ is the same for all values of i in question.