Solutions to Selected Exercises, HW #5

Assignment.

Chapter 4 problems, starting on p. 175: Problems 1, 2, 7bc, 8bc, 10, 20, 21, 25, 35a.

Exercise 1: Two balls are chosen randomly from an urn containing 8 white, 4 black, and 2 orange balls. Suppose that we win \$2 for each black ball selected and we lose \$1 for each white ball selected. Let X denote our winnings. What are the possible values of X, and what are the probabilities associated with each value?

SOLUTION: In dollars, X can be -2 (if two white balls are selected), -1 (if one white ball and one orange ball are selected); 0 (if two orange balls are selected); 1 (if one black ball and one white ball are selected); 2 (if one black ball and one orange ball are selected); or 4 (if two black balls are selected) . We have

$$\begin{aligned}
P(X=-2) &= \frac{8}{14}\cdot\frac{7}{13} = \frac{28}{91};\\
P(X=-1) &= \frac{2}{14}\cdot\frac{8}{13} + \frac{8}{14}\cdot\frac{2}{13} = \frac{16}{91}\\
P(X=0) &= \frac{2}{14}\cdot\frac{1}{13} = \frac{1}{91};\\
P(X=1) &= \frac{4}{14}\cdot\frac{8}{13} + \frac{8}{14}\cdot\frac{4}{13} = \frac{32}{91};\\
P(X=2) &= \frac{4}{14}\cdot\frac{2}{13} + \frac{2}{14}\cdot\frac{4}{13} = \frac{8}{91};\\
P(X=2) &= \frac{4}{14}\cdot\frac{3}{13} = \frac{6}{91}.
\end{aligned}$$

Note that the probabilities add up to $91/91 = 1$.

Exercise 7: Suppose that a die is rolled twice. What are the possible values that the following random variables can take on:

(b) the minimum value to appear in the two rolls;

(c) the sum of the two rolls.

SOLUTION: The minimum can equal 1, 2, 3, 4, 5, or 6. The sum can equal any integer from 2 to 12.

Exercise 8: If the die in Problem 4.7 is assumed fair, calculate the probabilities associated with the random variables in parts (a) through (d).

SOLUTION: **(b)** Of the 36 possible outcomes, only the outcomes 11, 12, 13, 14, 15, 16, 61, 51, 41, 31, 21 have a minimum of 1. So $P(\text{minimum equals } 1) = 11/36$. Similarly, $P(\text{minimum equals } 2) = 9/36$ (corresponding to outcomes 22, 23, 24, 25, 26, 62, 52, 42, 32), $P(\text{minimum equals } 3) = 7/36$, $P(\text{minimum equals } 4) = 5/36$, $P(\text{minimum equals } 5) = 3/36$, $P(\text{minimum equals } 6) = 1/36$. Note that these probabilities add up to $36/36 = 1$.

(c) We've done this in class. For example, $P(\text{sum equals } 5) = 4/36$, corresponding to the outcomes $14, 23, 32, 41$.

Exercise 10: Let X be the winnings of a gambler. Let $p(i) = P(X = i)$ and suppose that

$$p(0) = 1/3; \qquad p(1) = p(-1) = 13/55;$$
$$p(2) = p(-2) = 1/11; \qquad p(3) = p(-3) = 1/165.$$

Compute the conditional probability that the gambler wins i, $i = 1, 2, 3$, given that he wins a positive amount.

SOLUTION: Just divide $p(i)$ by $p(1) + p(2) + p(3)$. For example,

$$P(\text{gambler wins } 2) = \frac{p(2)}{p(1) + p(2) + p(3)} = \frac{1/11}{13/55 + 1/11 + 1/165} = \frac{3}{11}.$$

Exercise 20: A gambling book recommends the following "winning strategy" for the game of roulette: Bet \$1 on red. If red appears (which has probability 18/38), then take the \$1 profit and quit. If red does not appear and you lose this bet (which has probability 20/38 of occurring), make additional \$1 bets on red on each of the next two spins of the roulette wheel and then quit. Let X denote your winnings when you quit.

(a) Find $P\{X > 0\}$.

(b) Are you convinced that the strategy is indeed a "winning" strategy? Explain your answer!

(c) Find $E[X]$.

SOLUTION: (a) As per the given hints,

$$P(X > 0) = P(X = 1) = \frac{18}{38} + \frac{20}{38} \cdot \frac{18}{38} \cdot \frac{18}{38} = 0.5918.$$

(b) No I'm not convinced. Even though the probability of winning ($X > 0$) is positive, by part (a), the expected winnings are not, by part (c) below.

(c)

$$\begin{aligned} E[X] &= -3 \cdot P(X = -3) - 1 \cdot P(X = -1) + 1 \cdot P(X = 1) \\ &= -3 \cdot \left(\frac{20}{38}\right)^3 - 1 \cdot \frac{20}{38}\left(\frac{20}{38} \cdot \frac{18}{38} + \frac{18}{38} \cdot \frac{20}{38}\right) + 0.5918 = -0.108. \end{aligned}$$

Exercise 21: Four buses carrying 148 students from the same school arrive at a football stadium. The buses carry, respectively, 40, 33, 25, and 50 students. One of the students is randomly selected. Let X denote the number of students who were on the bus carrying the randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on her bus.

(a) Which of $E[X]$ or $E[Y]$ do you think is larger? Why?

(b) Compute $E[X]$ and $E[Y]$.

SOLUTION: (a) I think $E[X]$ is larger because, by selecting a student at random, there's a larger chance of selecting a student on a bus with more students.

(b)

$$
\begin{aligned}
E[X] &= 40\,P(X=40) + 33\,P(X=33) + 25\,P(X=25) + 50\,P(X=50) \\
&= 40 \cdot \frac{40}{148} + 33 \cdot \frac{33}{148} + 25 \cdot \frac{25}{148} + 50 \cdot \frac{50}{148} = 39.2838; \\
E[Y] &= 40\,P(Y=40) + 33\,P(Y=33) + 25\,P(Y=25) + 50\,P(Y=50) \\
&= 40 \cdot \frac{1}{4} + 33 \cdot \frac{1}{4} + 25 \cdot \frac{1}{4} + 50 \cdot \frac{1}{4} = 37.
\end{aligned}
$$

Exercise 25: Two coins are to be flipped. The first coin will land on heads with probability .6, the second with probability .7. Assume that the results of the flips are independent, and let X equal the total number of heads that result.

(a) Find $P\{X = 1\}$.

(b) Determine $E[X]$.

SOLUTION: (a)

$$
\begin{aligned}
&P(X = 1) \\
&= P(\text{first is heads and second is tails}) + P(\text{first is tails and second is heads}) \\
&= 0.6 \cdot (1 - 0.7) + (1 - 0.6) \cdot (0.7) = 0.46.
\end{aligned}
$$

(b)

$$
\begin{aligned}
E[X] &= 0 \cdot P(X=0) + 1 \cdot P(X=1) + 2 \cdot P(X=2) \\
&= 0 + 1 \cdot 0.46 + 2 \cdot 0.6 \cdot 0.7 = 1.3.
\end{aligned}
$$

Exercise 35a: A box contains 5 red and 5 blue marbles. Two marbles are withdrawn randomly. If they are the same color, then you win \$1.10; if they are different colors, then you win -\$1.00. (That is, you lose \$1.00.) Calculate

(a) the expected value of the amount you win.

SOLUTION: (a) Let X be the amount won in dollars. Then

$$\begin{aligned} E[X] &= \sum_{\substack{\text{values } x \\ \text{of } X}} x \cdot P(X = x) \\ &= 1.10 \cdot P(X = 1.10) - 1 \cdot P(X = 1) \\ &= 1.10 \cdot \left(\frac{5}{10} \cdot \frac{4}{9} + \frac{5}{10} \cdot \frac{4}{9} \right) - 1 \cdot \left(\frac{5}{10} \cdot \frac{5}{9} + \frac{5}{10} \cdot \frac{5}{9} \right) \\ &= -\frac{1}{15} = -0.0\overline{6}. \end{aligned}$$