Solutions to Selected Exercises, HW #4

Assignment.

Chapter 3, pages 103–111: Problems 1, 2, 5, 9, 12, 14, 16, 23.

Exercise 2: If two fair dice are rolled, what is the conditional probability that the first one lands on 6 given that the sum of the dice is i? Compute for all values of i between 2 and 12.

SOLUTION:

$$P(\text{first one lands on 6} \mid \text{sum on the dice is 2})$$
$$= \frac{|\text{first one lands on 6 and sum on the dice is 2}|}{|\text{sum on the dice is 2}|} = \frac{|\emptyset|}{|\{11\}|} = 0,$$
$$P(\text{first one lands on 6} \mid \text{sum on the dice is 3}) = \frac{|\emptyset|}{|\{12,21\}|} = 0,$$
$$\ldots$$
$$P(\text{first one lands on 6} \mid \text{sum on the dice is 7}) = \frac{|\{61\}|}{|\{16,25,34,43,52,61\}|} = \frac{1}{6},$$
$$P(\text{first one lands on 6} \mid \text{sum on the dice is 8}) = \frac{|\{62\}|}{|\{26,35,44,53,62\}|} = \frac{1}{5},$$
$$P(\text{first one lands on 6} \mid \text{sum on the dice is 9}) = \frac{|\{63\}|}{|\{36,45,54,63\}|} = \frac{1}{4},$$
$$\ldots$$
$$P(\text{first one lands on 6} \mid \text{sum on the dice is 12}) = \frac{|\{66\}|}{|\{66\}|} = 1.$$

Exercise 5: An urn contains 6 white and 9 black balls. If 4 balls are to be randomly selected without replacement, what is the probability that the first 2 selected are white and last 2 black?

SOLUTION: Let E be the event that the second two are black and F the event that the first two are white. By equation (2.2), page 59,

$$P(EF) = P(F) \cdot P(E|F).$$

Now since all outcomes are equally likely (because the balls are chosen randomly), and because there are originally 6 white and 9 black balls, we have

$$P(F) = \frac{\text{number of ways of choosing 2 balls from 6}}{\text{number of ways of choosing 2 balls from 15}} = \frac{\binom{6}{2}}{\binom{15}{2}}.$$

Next, given that the first two balls are white, there are 13 balls remaining of which 9 are black, so

$$P(E|F) = \frac{\text{number of ways of choosing 2 balls from 9}}{\text{number of ways of choosing 2 balls from 13}} = \frac{\binom{9}{2}}{\binom{13}{2}}.$$

So

$$P(EF) = \frac{\binom{6}{2}}{\binom{15}{2}} \cdot \frac{\binom{9}{2}}{\binom{13}{2}} = \frac{6}{91} = 0.0659341.$$

Exercise 9: Consider 3 urns. Urn A contains 2 white and 4 red balls, urn B contains 8 white and 4 red balls, and urn C contains 1 white and 3 red balls. If 1 ball is selected from each urn, what is the probability that the ball chosen from urn A was white given that exactly 2 white balls were selected?

SOLUTION: Let A be the event that the ball selected from urn A is white, B the event that the ball selected from urn B is white, and C the event that the ball selected from urn C is white. Let T be the event that exactly two of the selected balls are white. Then

$$P(A|T) = \frac{P(AT)}{P(T)}.$$

Now note that saying *exactly* two balls are white amounts to saying that either A and B happen but C doesn't, or A and C happen but B doesn't, or B and C happen but A doesn't. So

$$T = ABC^c \cup AB^cC \cup A^cBC.$$

Similarly,

$$AT = ABC^c \cup AB^cC.$$

Since ABC^c, AB^cC, and A^cBC are mutually exclusive (why?), and A, B, and C are independent, we find that

$$P(A|T) = \frac{P(ABC^c \cup AB^cC)}{P(ABC^c \cup AB^cC \cup A^cBC)} = \frac{\frac{2}{6}\frac{8}{12}\frac{3}{4} + \frac{2}{6}\frac{4}{12}\frac{1}{4}}{\frac{2}{6}\frac{8}{12}\frac{3}{4} + \frac{2}{6}\frac{4}{12}\frac{1}{4} + \frac{4}{6}\frac{8}{12}\frac{1}{4}}$$

$$= \frac{\frac{2}{6}\frac{8}{12}\frac{3}{4} + \frac{2}{6}\frac{4}{12}\frac{1}{4}}{\frac{2}{6}\frac{8}{12}\frac{3}{4} + \frac{2}{6}\frac{4}{12}\frac{1}{4} + \frac{4}{6}\frac{8}{12}\frac{1}{4}} = \frac{7}{11} = 0.636364.$$

Exercise 12: Suppose distinct values are written on each of 3 cards, which are then randomly given the designations A, B, and C. Given that card A's value is less than card B's value, find the probability it is also less than card C's value.

SOLUTION: List the cards in order of ascending value. For example, if the value of card C is smallest, followed by the value of card A, followed by the value of card B, then write CAB. Then

$$P(A < C|A < B) = \frac{P(A < C \text{ and } A < B)}{P(A < B)} = \frac{|\{ABC, ACB\}|}{|\{ABC, ACB, CAB\}|} = \frac{2}{3} = 66.\overline{6}\%.$$

Exercise 14: Suppose that an ordinary deck of 52 cards (which contains 4 aces) is randomly divided into 4 hands of 13 cards each. We are interested in determining

p, the probability that each hand has an ace. Let E_i be the event that the ith hand has exactly one ace. Determine $p = P(E_1E_2E_3E_4)$ by using the multiplication rule.

SOLUTION: We have

$$P(E_1E_2E_3E_4) = P(E_1)P(E_2|E_1)P(E_3|E_1E_2)P(E_4|E_1E_2E_3).$$

To compute $P(E_1)$, first note that there are $\binom{52}{13}$ choices for the first 13-card hand. How many of these choices have exactly one ace? Well, there are 4 ways of choosing which ace, and for each such choice, there are $\binom{48}{12}$ choices for the remaining 12 cards (since the remaining 12 all have to be non-aces).

To compute $P(E_2|E_1)$ note that, if E_1 has been achieved, then there are $\binom{39}{13}$ choices for the second 13-card hand, of which $3 \cdot \binom{36}{12}$ have exactly one ace (there are three choices left for which ace, and then the remaining 12 cards can be neither an ace nor one of the other 12 cards chosen for the first hand, so there are $52 - 4 - 12$ cards from which to choose).

Continuing in this way, we find that

$$p = \frac{4 \cdot \binom{48}{12}}{\binom{52}{13}} \cdot \frac{3 \cdot \binom{36}{12}}{\binom{39}{13}} \cdot \frac{2 \cdot \binom{24}{12}}{\binom{26}{13}} \cdot \frac{1 \cdot \binom{12}{12}}{\binom{13}{13}} = 0.105498.$$

Exercise 16: An ectopic pregnancy is twice as likely to develop when the pregnant woman is a smoker as it is when she is a nonsmoker. If 32 percent of women of childbearing age are smokers, what percentage of women having ectopic pregnancies are smokers?

SOLUTION: Let S and E be the obvious events here. We're trying to find $P(S|E)$.

Note that, by equation (2.1),

$$P(S|E) = \frac{P(SE)}{P(E)}. \tag{1}$$

But by equation (2.2),

$$P(ES) = P(S)P(E|S). \tag{2}$$

Since ES is the same as SE, we can plug (2) into (1) to get

$$P(S|E) = \frac{P(S)P(E|S)}{P(E)}. \tag{3}$$

We are given $P(S)$, so if we can figure out what $P(E|S)/P(E)$ is, we're done.

To figure this out note that, by equation (3.1) on page 64, also known as Bayes's formula,

$$P(E) = P(E|S)P(S) + P(E|S^c)(1 - P(S)).$$

Into this formula, plug in the given facts $P(S) = .32$ and $P(E|S^c) = P(E|S)/2$, to get

$$P(E) = P(E|S) \cdot .32 + \tfrac{1}{2}P(E|S) \cdot (1 - .32) = .32P(E|S) + .34P(E|S) = .66P(E|S).$$

So $P(E|S)/P(E) = 1/.66$. So by (3) above,

$$P(S|E) = \frac{P(S)P(E|S)}{P(E)} = \frac{.32}{.66} = 0.484848.$$

Exercise 23: A red die, a blue die, and a yellow die (all six sided) are rolled. We are interested in the probability that the number appearing on the blue die is less than that appearing on the yellow die, which is less than that appearing on the red die. That is, with B, Y, and R denoting, respectively, the number appearing on the blue, yellow, and red die, we are interested in $\mathrm{P}(B < Y < R)$.

(a) What is the probability that no two of the dice land on the same number? **(b)** Given that no two of the dice land on the same number, what is the conditional probability that $B < Y < R$? **(c)** What is $P(B < Y < R)$?

SOLUTION:

The sample space has size 6^3.

(a) There are 6 possibilities for the blue die, then 5 for the yellow, then 4 for the red. So

$$P(\text{no two numbers are the same}) = \frac{6 \cdot 5 \cdot 4}{6^3} = \frac{5}{9}.$$

(b) Writing BYR for the outcome where $B < Y < R$, and so on, we have

$$P(B < Y < R \text{ given that no two are the same}) = \frac{|\{BYR\}|}{|\{BYR, BRY, YBR, YRB, RBY, RYB\}|} = \frac{1}{6}.$$

(c)

$P(B < Y < R)$
$= P(\text{no two numbers are the same}) \cdot P(B < Y < R \text{ given that no two are the same})$
$= \frac{5}{9} \cdot \frac{1}{6} = \frac{5}{54} = 0.0925926.$