Solutions to Selected Exercises, HW #3

Assignment.

Chapter 2, pages 50–54: 8, 9, 10, 12, 15, 16, 25, 27, 41.

Exercise 8. Suppose that A and B are mutually exclusive events for which $P(A) = .3$ and $P(B) = .5$. What is the probability that

(a) either A or B occurs?

(b) A occurs but B does not?

(c) both A and B occur?

SOLUTION: **(a)** $P(A \cup B) = P(A) + P(B) = .3 + .5 = .8$.

(b) A and B are mutually exclusive so they cannot both happen together, so to say A happens is exactly the same as saying A and B^c happen. So $P(AB^c) = P(A) = .3$

(c) A and B are mutually exclusive so they cannot both happen together, so $P(AB) = 0$.

Exercise 9. A retail establishment accepts either the American Express or the VISA credit card. A total of 24 percent of its customers carry an American Express card, 61 percent carry a VISA card, and 11 percent carry both cards. What percentage of its customers carry a credit card that the establishment will accept?

SOLUTION: If A and V denote the obvious events in this context, then you are given $P(A) = .24$, $P(V) = .61$, and $P(AV) = .11$. So $P(A \cup V) = P(A) + P(V) - P(AV) = .24 + .61 - .11 = .74$, or 74 percent.

Exercise 10: Sixty percent of the students at a certain school wear neither a ring nor a necklace. Twenty percent wear a ring and 30 percent wear a necklace. If one of the students is chosen randomly, what is the probability that this student is wearing

(a) a ring or a necklace?

(b) a ring and a necklace?

SOLUTION: If R and N denote the obvious events in this context, then you are given $P(R) = .2$, $P(N) = .3$, and $P((R \cup N)^c) = .6$. From the latter fact, we conclude that

(a) $P(R \cup N) = 1 - .6 = .4$. So

(b) $P(RN) = P(R) + P(N) - P(RN) = .2 + .3 - .4 = .1$.

Exercise 12: An elementary school is offering 3 language classes: one in Spanish, one in French, and one in German. The classes are open to any of the 100 students in the school. There are 28 students in the Spanish class, 26 in the French class, and 16 in the German class. There are 12 students who are in both Spanish and French, 4 who are in both Spanish and German, and 6 who are in both French and German. In addition, there are 2 students taking all 3 classes.

(a) If a student is chosen randomly, what is the probability that he or she is not in any of the language classes?

(b) If a student is chosen randomly, what is the probability that he or she is taking exactly one language class?

(c) If 2 students are chosen randomly, what is the probability that at least 1 is taking a language class?

SOLUTION: Let S, F, and G denote the obvious events.

(a) We first compute $P(S \cup F \cup G)$:

$$\begin{aligned} P(S \cup F \cup G) &= P(S) + P(F) + P(G) - P(SF) - P(SG) - P(FG) + P(SFG) \\ &= .28 + .26 + .16 - .12 - .04 - .06 + .02 = .5. \end{aligned}$$

Not taking any classes is the complement of taking at least one, so $P(S^c F^c G^c) = 1. - 5 = .5$.

(b) There are several ways to approach this; here's one. The number taking exactly one is the number taking at least one, minus the number taking exactly two, minus the number taking exactly three. The number taking at least one is 50 (since, as we computed in part (a), $P(S \cup F \cup G) = .5$). The number taking Spanish and French *only* is 10 (12 in Spanish and French minus 2 who are taking all three). Similarly, the number taking Spanish and German only is $4 - 2 = 2$, and the number taking French and German only is $6 - 2 = 4$. The number taking all three is 2. So the number taking exactly one is

$$50 - 10 - 2 - 4 - 2 = 32,$$

so the probability of a randomly selected student taking exactly one is $32/100 = .32$.

(c) Let A be the event that the first student is taking at least one, and B the event that the second student is. Then the probability in question is

$$P(A \cup B) = P(A) + P(B) - P(AB).$$

We have $P(A) = P(B) = P(S \cup F \cup G) = .5$ by part (a). To compute $P(AB)$, we note the following. There are $\binom{100}{2}$ ways of choosing two students out of 100, and $\binom{50}{2}$ ways of picking 2 out of the 50 who are taking at least one class. So $P(AB) = \binom{50}{2} / \binom{100}{2} = 0.247475$. So

$$P(A \cup B) = P(A) + P(B) - P(AB) = .5 + .5 - .247475 = .752525.$$

Exercise 15: 15. If it is assumed that all $\binom{52}{5}$ poker hands are equally likely, what is the probability of being dealt

(a) a flush? (A hand is said to be a flush if all 5 cards are of the same suit.)

(b) one pair? (This occurs when the cards have denominations a, a, b, c, d, where a, b, c, and d are all distinct.)

(c) two pairs? (This occurs when the cards have denominations a, a, b, b, c, where a, b, and c are all distinct.)

(d) three of a kind? (This occurs when the cards have denominations a, a, a, b, c, where a, b, and c are all distinct.)

(e) four of a kind? (This occurs when the cards have denominations a, a, a, a, b.)

SOLUTION: There are $\binom{52}{5}$ possble 5-card hands. **(a)** To form a flush, we first pick a suit (there are $\binom{4}{1}$ ways); then pick 5 different cards from that suit ($\binom{13}{5}$ ways). So

$$P(\text{flush}) = \frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}} = 0.00198079.$$

(b) Choose the face value for the pair: there are 13 ways to do this. Then choose 2 cards out of the 4 with the given face value: there are $\binom{4}{2}$ ways to do this. Of the remaining 12 face values, choose 3: there are $\binom{12}{3}$ ways. Then choose one card for each of these three face values: there are $4 \cdot 4 \cdot 4$ ways. So

$$P(\text{one pair}) = \frac{13 \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot 4 \cdot 4 \cdot 4}{\binom{52}{5}} = 0.422569.$$

(c) Choose the two face values for the two pairs: there are $\binom{13}{2}$ ways to do this. Then choose 2 cards out of the 4 with each face value: there are $\binom{4}{2} \cdot \binom{4}{2}$ ways to do this. Of the remaining 11 face values, choose 1: there are 11 ways. Then choose one card of this face value: there are 4 ways. So

$$P(\text{two pairs}) = \frac{\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 11 \cdot 4}{\binom{52}{5}} = 0.047539.$$

(d)

$$P(\text{three of a kind}) = \frac{13 \cdot \binom{4}{3} \cdot \binom{12}{2} \cdot 4 \cdot 4}{\binom{52}{5}} = 0.0211285.$$

(e)

$$P(\text{four of a kind}) = \frac{\binom{13}{1}\binom{48}{1}}{\binom{52}{5}} = 0.000240096.$$

Exercise 16: Poker dice is played by simultaneously rolling 5 dice. Show that

(a) P(no two alike) = .0926;

(b) P(one pair) = .4630;

(c) P(two pair) = .2315;

(d) P(three alike) = .1543;

(e) P(full house) = .0386;

(f) P(four alike) = .0193;

(g) P(five alike) = .0008.

SOLUTION: Count the outcomes in each event, and then divide by 6^5. For example, for part (b), pick the 2 slots out of 5 where you want the matching pair to be: there are $\binom{5}{2}$ ways to do this, and for each of these ways, there are 6 choices for what number is showing for this pair. Now fill in the remaining three slots: there are $5 \cdot 4 \cdot 3$ ways to do this. So

$$P(\text{one pair}) = \frac{\binom{5}{2} \cdot 6 \cdot 5 \cdot 4 \cdot 3}{6^5} = 0.462963.$$

Similarly (think it through on your own),

$$P(\text{full house}) = \frac{\binom{5}{2} \cdot 6 \cdot 5}{6^5} = 0.0385802.$$

Exercise 25: A pair of dice is rolled until either a sum of 5 or 7 appears. Find the probability that a 5 occurs first.

SOLUTION: From all the hints given, you should be able to conclude that

$$\begin{aligned} P(\text{a sum of 5 occurs first}) &= \sum_{n=1}^{\infty} P(E_n) = \sum_{n=1}^{\infty} \frac{4}{36} \cdot \left(\frac{26}{36}\right)^{n-1} = \frac{4}{36} \sum_{n=1}^{\infty} \left(\frac{26}{36}\right)^{n-1} \\ &= \frac{4}{36} \cdot \frac{1}{1 - 26/36} = \frac{4}{36 - 26} = \frac{4}{10} = .4. \end{aligned}$$

Exercise 27: An urn contains 3 red and 7 black balls. Players A and B withdraw balls from the urn consecutively until a red ball is selected. Find the probability that A selects the red ball. (A draws the first ball, then B, and so on. There is no replacement of the balls drawn.)

SOLUTION:

From the hints, you should get

$$\begin{aligned} P(A \text{ selects the red ball}) &= \frac{3}{10} + \frac{7 \cdot 6 \cdot 3}{10 \cdot 9 \cdot 8} + \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6} + \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 3}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4} \\ &= 0.583333. \end{aligned}$$

Exercise 41 If a die is rolled 4 times, what's the probability that 6 comes up at least once?

SOLUTION:

$$P(\text{at least one } 6) = 1 - P(\text{no } 6\text{'s}) = 1 - \frac{5^4}{6^4} = 0.517747.$$