Hints for Selected Exercises, HW #3

Assignment.

Chapter 2, pages 50–54: 8, 9, 10, 12, 15, 16, 25, 27, 41.

Exercise 8: **(b)** $P(AB^c) = P(A)$. To get full credit, *please explain why this is so.* **(c)** You don't need a formula. Just think about what "mutually exclusive" means.

Exercise 9: See Proposition 4.3.

Exercise 10: If R and N denote the obvious events in this context, then you are give $P(R)$, $P(N)$, and $P((R \cup N)^c)$. This should be enough, using Propositions 4.1 and 4.3.

Exercise 12: See Proposition 4.4.

Exercise 15: We did some of the combinatorics needed for this problem in class. See the lecture notes from Friday, 8/30.

Exercise 16: The sample space has size 6^5 (explain why). **(a)** The first die can be anything; that is, there are six ways for the first die to land. But then there are only 5 ways the second die can land (because it can't be the same as the first), and then 4 ways for the third die, and so on.

Exercise 25: This one is quite tricky! Let's think it through.

First of all, the book is a bit ambiguous in its wording. Note that the problem is about which SUMS show up. ("A pair of dice is rolled until a sum of either 5 or 7 appears.") So when the book says "Find the probability that a 5 occurs first," it means "Find the probability that a SUM of 5 occurs first." And when it says "Let E_n denote the event that a 5 occurs on the nth roll and no 5 or 7 occurs on the first $n-1$ rolls," it means "Let E_n denote the event that a SUM of 5 occurs on the nth roll and no SUM of 5 or 7 occurs on the first $n-1$ rolls."

Given this, let's find $P(E_n)$. If we roll the two dice a *single* time, then there are $6^2 = 36$ outcomes. So if we roll n times, there are 36^n possible outcomes.

In a single roll, 4 outcomes sum to 5 (14, 23, 32, and 41), while 6 sum to 7 (16, 25, 34, 43, 52, 61). So there are $36 - 4 - 6 = 26$ outcomes that don't sum to either a 5 or a 7. Therefore, the number of ways that the first $n-1$ rolls sum to neither 5 nor 7, and the nth roll sums to 5, is $26^{n-1} \cdot 4$.

So $P(E_n) = (26^{n-1} \cdot 4)/36^n$. Noting that E_1, E_2, $E_3, \ldots$ are mutually exclusive, we find that

$$P(\text{a sum of 5 occurs first}) = \sum_{n=1}^{\infty} P(E_n) = \sum_{n=1}^{\infty} \frac{26^{n-1} \cdot 4}{36^n}.$$

(Explain why the above formula is true.) To finish this problem, it may help to note that:

(a) First of all,

$$\frac{26^{n-1} \cdot 4}{36^n} = \left(\frac{26}{36}\right)^{n-1} \cdot \frac{4}{36};$$

(b) For any number r with $|r| < 1$,

$$\sum_{n=1}^{\infty} r^{n-1} = \frac{1}{1-r}.$$

Make sure you get the same answer as in the back of the book.

Exercise 27: First of all, this game can last at most 4 rounds. Why? Well if 3 rounds go by then, since both A and B draw on each round, this means that 6 draws took place. And if the game has not ended by the end of the third round, this means that all 6 of those draws were black. And since balls are not replaced, this means there's only one black ball left by the fourth round. Either A draws the red ball in the fourth round, in which case the game is over, or A draws the last black ball, in which case B will draw the red, and the game is over.

So we only need compute $P(E_n)$ for $1 \leq n \leq 4$, where E_n is the event where A draws the red ball in round n (and not before).

We have $P(E_1) = 3/10$. Explain why.

What is $P(E_2)$? Well to say A drew the red ball in round 2 is to say the two draws (A and B) in round 1 were black, and the third (A) was red. It follows that $P(E_2) = (7 \cdot 6 \cdot 3)/(10 \cdot 9 \cdot 8)$. Explain why.

Similarly, $P(E_3) = (7 \cdot 6 \cdot 5 \cdot 4 \cdot 3)/(10 \cdot 9 \cdot 8 \cdot 7 \cdot 6)$. Explain why.

Finally, $P(E_4) = (7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 3)/(10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4)$. Explain why.

Now add everything up. Of course, you should get a number between 0 and 1.

Exercise 41: **(a)** It's easier to first compute the probability that no 6 comes up.