Solutions to Selected Exercises, HW #2

Assignment.

- Chapter 1, pages 15-17: Problems 21, 25, 30, 31, 32.
- Chapter 2, pages 50-54: Problems 1, 3, 5, 6, 7.

Chapter 1

Exercise 21: From a group of 8 women and 6 men, a committee consisting of 3 men and 3 women is to be formed. How many different committees are possible if

(a) 2 of the men refuse to serve together?

(b) 2 of the women refuse to serve together?

(c) 1 man and 1 woman refuse to serve together?

SOLUTION: (a) There are $\binom{8}{3}$ choices for the 3 women. For the 3 men, there are $\binom{4}{2}$ choices of the 3 men if we exclude one of the two who refuse to work together but include the other (because by excluding one and including the other, we only need to choose 2 more of the remaining 4). Similarly, there are $\binom{4}{2}$ choices if we exclude the other of those two men but not the first, and there are $\binom{4}{3}$ choices if we exclude both. This gives a total of

$$\binom{8}{3}\left(\binom{4}{2}+\binom{4}{2}+\binom{4}{3}\right)=896$$

possible committees.

Another way to see this is: in general, there are $\binom{8}{3}\binom{6}{3}$ ways to form a committee of 3 women and 3 men from the 8 women and 6 men. But if two men refuse to work together, we must subtract the number of committees that contain both of the men; this is $\binom{8}{3}\binom{4}{1}$ (if both men are on the committee, there is only one more member to choose from the remaining 4 men). This way of counting yields

$$\binom{8}{3}\binom{6}{3}-\binom{8}{3}\binom{4}{1}=896$$

committees, again.

(b) In this case, by a similar argument, there are

$$\binom{6}{3}\left(\binom{6}{2}+\binom{6}{2}+\binom{6}{3}\right)=1{,}000$$

possible committees. (Or: $\binom{6}{3}\binom{8}{3}-\binom{6}{3}\binom{6}{1}=1{,}000$.)

(c) There are $\binom{7}{2}\binom{5}{3}$ committees where the woman serves but the man doesn't, $\binom{7}{3}\binom{5}{2}$ where the woman doesn't serve but the man does, and $\binom{7}{3}\binom{5}{3}$ where neither serves, giving

$$\binom{7}{2}\binom{5}{3} + \binom{7}{3}\binom{5}{2} + \binom{7}{3}\binom{5}{3} = 910$$

possible committees total. (Or: $\binom{8}{3}\binom{6}{3} - \binom{7}{2}\binom{5}{2} = 910$.)

Exercise 25: A psychology laboratory conducting dream research contains rooms, with 2 beds in each room. If 3 sets of identical twins are to be assigned to these s6 beds so that each set of twins sleeps in different beds in the same room, how many assignments are possible?

SOLUTION: First assign a room to each set of twins; there are 3! ways of doing this. (3 choices for the first set, leaving two choices for the second set, leaving 1 choice for the third.). Then within each of the three rooms, there are to choices for which twin gets which bed. So there are

$$3! \cdot 2 \cdot 2 \cdot 2 = 48$$

possible assignments.

Exercise 30: If 12 people are to be divided into 3 committees of respective sizes 3, 4, and 5, how many divisions are possible?

SOLUTION: By Section 1.5 on multinomial coefficients, the answer is

$$\binom{12}{3,4,5} = \frac{12!}{3!4!5!} = 27{,}720.$$

Exercise 31: If 8 new teachers are to be divided among 4 schools, how many divisions are possible? What if each school must receive 2 teachers?

SOLUTION: In the first case, we argue as follows. There are 4 choices for where to put each teacher, yielding

$$4^8 = 65{,}536$$

possibilities.

If each school must receive 2 teachers, then the answer is

$$\binom{8}{2,2,2,2} = 2{,}520$$

possibilities.

Exercise 32: Ten weight lifters are competing in a team weight-lifting contest. Of the lifters, 3 are from the United States, 4 are from Russia, 2 are from China, and 1 is from Canada. If the scoring takes account of the countries that the lifters represent, but not their individual identities, how many different outcomes are possible from the point of view of scores? How many different outcomes correspond to results in which the United States has 1 competitor in the top three and 2 in the bottom three?

SOLUTION: Imagine that the 10 (3+4+2+1) scores are listed from highest to lowest. Since the scores reflect only the nationality, we can think of the 3 Unites States scores as being the same as each other, and similarly for the 4 Russian scores, 2 scores from China, and 1 from Canada. So we are arranging 10 objects, where 3, 4, 2, and 1 of these, respectively, are identical, yielding

$$\frac{10!}{3!4!2!1!} = 12{,}600$$

possibilities.

Now, suppose the United States has 1 competitor in the top three and 2 in the bottom three. There are 3 possibilities for where the top-three competitor places: first, second, or third place. Similarly, there are $\binom{3}{2} = 3$ possibilities for where the two in the bottom three place. There are 7 remaining competitors, divided into groups of 4, 2, and 1, so there are

$$\frac{7!}{4!2!1!} = 105$$

ways that these 7 can place. So all told, there are

$$3 \cdot 3 \cdot 105 = 945$$

possible placements.

Chapter 2

Exercise 1: A box contains 3 marbles: 1 red, one green, and one blue. Consider an experiment that consists of taking one marble from the box and then replacing it in the box and then drawing a second marble from the box. Describe the sample space. Repeat when the second marble is drawn without replacing the first marble.

SOLUTION: The first scenario is called *sampling with replacement.* In this case the sample space S might look like

$$\begin{aligned} S &= \{\text{two-letter strings: each letter is an } R,\, G, \text{ or } B\} \\ &= \{RR, RG, RB, GR, GG, GB, BR, BG, BB\}. \end{aligned}$$

In the second scenario, we are *sampling without replacement*, and the sample space S might be described as

$$\begin{aligned} S &= \{\text{strings of two } \textit{distinct} \text{ letters: each letter is an } R,\, G, \text{ or } B\} \\ &= \{RG, RB, GR, GB, BR, BG\}. \end{aligned}$$

Exercise 3: Two dice are thrown. Let E be the event that the sum of the dice is odd, let F be the event that at least one of the dice lands on 1, and let G be the event that the sum is 5. Describe the events EF, $E \cup F$, FG, EF^c, and EFG.

SOLUTION: Describe an outcome as a two-digit string, where the first digit is how the first die lands and the second is how the second lands. Then:

$$\begin{aligned}
EF &= \text{the event where the sum is odd } \textit{and} \text{ at least one die lands on a 1} \\
&= \{12{,}14{,}16{,}21{,}41{,}61\}; \\
E \cup F &= \text{the event where the sum is odd } \textit{or} \text{ at least one die lands on a 1} \\
&= \{11{,}12{,}13{,}14{,}15{,}16{,}21{,}31{,}41{,}51{,}61{,}23{,}25{,}32{,}34{,}36{,}43{,}45{,}52{,}54{,}56{,}63{,}65\}; \\
FG &= \text{the event where at least one die lands on a 1 and the sum is 5} \\
&= \{14{,}41\}; \\
EF^c &= \text{the event where the sum is odd and neither one lands on a 1} \\
&= \{23{,}25{,}32{,}34{,}36{,}43{,}45{,}52{,}54{,}56{,}63{,}65\}; \\
EFG &= \text{the event where the sum is odd and at least one die lands on a 1 and} \\
&\quad \text{the sum is } 5 = \{14{,}41\}.
\end{aligned}$$

Exercise 6: A hospital administrator codes incoming patients suffering gunshot wounds according to whether they have insurance (coding 1 if they do an 0 if they do not) and 0 if they do not) and according to their condition, which is rated as good (g), fair (f), or serious (s). Consider an experiment that consists of the coding of such a patient.

(a) Give the sample space of this experiment.

(b) Let A be the event that the patient is in serious condition. Specify the outcomes in A.

(c) Let B be the event that the patient is uninsured. Specify the outcomes in B.

(d) Give all the outcomes in the event $B^c \cup A$.

SOLUTION: **(a)** $S = \{0g{,}0f{,}0s{,}1g{,}1f{,}1s\}$. **(b)** $A = \{0s{,}1s\}$. **(c)** $B = \{0g{,}0f{,}0s\}$. **(d)** $B^c \cup A = \{0s{,}1g{,}1f{,}1s\}$.