1. A certain basketball player takes 30 shots in a game. Suppose the player is a 60% shooter (meaning 60% of their shots go in). What is the expected number of times that this player will hit two shots in a row, in such a game? Hint: let X_1 equal 1 if the first two shots go in, and 0 otherwise. Let X_2 equal 1 if the second and third shots go in, and 0 otherwise. And so on. Then let $X = X_1 + X_2 + ...$, and compute $E(X)$.

   (We assume independence of the shots.) If $X_1, X_2, X_3, \ldots, X_{29}$ are as above, then each X_i is a Bernoulli random variable with $p = P(\text{success}) = 0.6^2 = 0.36$. Also, if X is the sum of the X_i's, then by the sum rule,

$$E[X] = \sum_{i=1}^{29} E[X_i] = \sum_{i=1}^{29} 0.36 = 29 \cdot 0.36 = 10.44.$$

2. You pay \$6 to play a game where a fair die is rolled. You lose if the die lands on an even number, you receive \$9 if the die lands on a 1 or a 3, and you receive \$12 if it lands on a 5.

   (a) Find the probability mass function for your payoff X (meaning how much you receive minus the \$6 put in to play).

   The possible values of the payoff X, in dollars, are -6 (if you lose), $9-6=3$ (if you roll a 1 or a 3), and $12-6=6$ (if you roll a 5). We have $P(X=-6) = P(\text{roll an even number}) = 1/2$, $P(X=3) = P(\text{roll a 1 or a 3}) = 1/3$, and $P(X=6) = P(\text{roll a 5}) = 1/6$.

   (b) What are your expected winnings (meaning how much you receive minus the \$6 put in to play) from this game?

$$E[X] = -6 \cdot P(X=-6) + 3 \cdot P(X=3) + 6 \cdot P(X=6) = -6 \cdot \frac{1}{2} + 3 \cdot \frac{1}{3} + 6 \cdot \frac{1}{6} = -1$$

   dollars.

3. You pay \$5 to play the following game. You toss an unfair coin, with P(heads)= 1/3. If the coin lands heads, you choose two marbles at random from a jar containing 3 red marbles and 2 blue marbles. If the coin lands tails, you choose two marbles from a jar containing 4 red marbles and 2 blue marbles. You are then awarded \$15 if you end up with two red marbles; otherwise, you receive \$0.

   (a) Find the probability mass function for your payoff X (meaning how much you receive minus the \$5 put in to play).

   The payoff X is either -5 dollars or 10 dollars. We have

$$\begin{aligned} P(X=10) &= P(\text{two red marbles}) \\ &= P(\text{coin lands heads}) \cdot P(\text{two red marbles given coin lands heads}) \\ &+ P(\text{coin lands tails}) \cdot P(\text{two red marbles given coin lands tails}) \\ &= \frac{1}{3} \cdot \frac{3}{5} \cdot \frac{2}{4} + \frac{2}{3} \cdot \frac{4}{6} \cdot \frac{3}{5} = \frac{11}{30}. \end{aligned}$$

Since $X = -5$ and $X = 10$ are the only two posibilities, we must have

$$P(X = -5) = 1 - P(X = 10) = 1 - \frac{11}{30} = \frac{19}{30}.$$

(b) What are your expected winnings (meaning how much you receive minus the \$5 put in to play) from this game?

$$E[X] = -5 \cdot P(X = -5) + 10P(X = 10) = -5 \cdot \frac{19}{30} + 10 \cdot \frac{11}{30} = \frac{-95 + 110}{30} = \frac{15}{30} = 0.5$$

dollars.

4. Here we consider two similar scenarios. For both of these, recall that a standard deck of cards has 4 aces, and 48 cards that aren't aces.

(A) We draw three cards *without replacement* at random from a standard deck. That is: each card that is drawn is left out of the deck before drawing the next card. (This is the usual way cards are drawn or dealt.) Let X be the number of aces drawn.

(B) We draw three cards *with replacement* at random from a standard deck. This means: We draw the first card, record what kind of card it is (ace or not), and *put it back in the deck*. We then draw the second card, record what kind of card it is (ace or not), and *put it back in the deck*. Finally, we draw the third card, and record what kind of card it is (ace or not). Let Y be the total number of aces drawn.

Here are the questions.

(a) One of the above scenarios represents a binomial experiment, and one does not. Which is which? Why?

Scenario (A) *does not* represent a binomial experiment, because the three draws of a card *are not* independent events. In scenario (A), since you're leaving cards out, the probability of getting an ace in the second draw will depend on whether you got one in the first, and the probability of getting an ace in the third draw will depend on what happened in the first two.

Scenario (B) *does* represent a binomial experiment, because the three draws of a card *are* independent events. In scenario (B), since you're replacing cards after drawing them, the probability of getting an ace in the second draw *will not* depend on whether you got one in the first, and the probability of getting an ace in the third draw *will not* depend on what happened in either of the first two.

(b) Compute the probability mass function for X (from scenario (A) above).

There are $\binom{52}{3}$ total ways of drawing three cards out of the 52. Out of all these ways, how many of these ways have *exactly* k aces? Well first note that, if there are exactly k aces, then there are exactly $3 - k$ non-aces. There are $\binom{4}{k}$ ways of drawing k aces out of the 4 aces, and there are $\binom{48}{3-k}$ ways of drawing $3-k$ non-aces out of the 48 non-aces. So the probability of getting k aces in three draws is

$$P(X=k) = \frac{\binom{4}{k}\cdot\binom{48}{3-k}}{\binom{52}{3}}.$$

So we compute:

$$P(X=0) = \frac{\binom{4}{0}\cdot\binom{48}{3}}{\binom{52}{3}} = 0.78262,$$
$$P(X=1) = \frac{\binom{4}{1}\cdot\binom{48}{2}}{\binom{52}{3}} = 0.20416,$$
$$P(X=2) = \frac{\binom{4}{2}\cdot\binom{48}{1}}{\binom{52}{3}} = 0.01303,$$
$$P(X=3) = \frac{\binom{4}{3}\cdot\binom{48}{0}}{\binom{52}{3}} = 0.00018.$$

(c) Compute $E[X]$.

$$\begin{aligned} E(X) &= 0\cdot P(X=0) + 1\cdot P(X=1) + 2\cdot P(X=2) + 3\cdot P(X=3) \\ &= 0\cdot 0.78262 + 1\cdot 0.20416 + 2\cdot 0.01303 + 3\cdot 0.00018 = 0.23076. \end{aligned}$$

(d) This time, compute the probability mass function for Y (from scenario (B) above).
Since this is a binomial experiment with three independent trials, and with $P(\text{success}) = 4/52 = 1/13$, we have

$$P(Y=k) = \binom{3}{k}\left(\frac{1}{13}\right)^k\left(\frac{12}{13}\right)^{3-k}.$$

So we compute:

$$P(Y=0) = \binom{3}{0}\left(\frac{1}{13}\right)^0\left(\frac{12}{13}\right)^3 = 0.78653,$$
$$P(Y=1) = \binom{3}{1}\left(\frac{1}{13}\right)^1\left(\frac{12}{13}\right)^2 = 0.19663,$$
$$P(Y=2) = \binom{3}{2}\left(\frac{1}{13}\right)^2\left(\frac{12}{13}\right)^1 = 0.01639,$$
$$P(Y=3) = \binom{3}{3}\left(\frac{1}{13}\right)^3\left(\frac{12}{13}\right)^0 = 0.00046.$$

(e) Compute $E[Y]$.

$$E(Y) = np = 3 \cdot \frac{1}{13} = 0.23077.$$

(f) How do $E[X]$ and $E[Y]$ compare?

They are nearly the same (maybe exactly the same, after accounting for roundoff error).

(g) Which is larger, $P(X = 0)$ or $P(Y = 0)$? Why does this make sense? That is, could you have predicted this without any computation, and if so, how?

$P(Y = 0) = 0.78653$, which is slightly larger than $P(X = 0) = 0.78262$. This makes sense: there are more non-aces than aces, so on average, when you put a card back into the deck, you are *decreasing* the probability of the next card being an ace. So on average, when you replace cards, you are *increasing* the probability of *not* getting aces. So scenario (b) will give a slightly higher probability of getting no aces than scenario (a), meaning $P(Y = 0) > P(X = 0)$.

(h) Compute $\text{Var}[X]$ and $\text{Var}[Y]$. How do they compare? Why does this make sense? That is, could you have predicted this without any computation, and if so, how?

$$\text{Var}[X] = \sum_{k=0}^{3} (k - \mu)^2 P(X = k) = 0.204664$$

(here, $\mu = E[X]$). Also,

$$\text{Var}[Y] = np(1 - p) = 3 \cdot \frac{1}{13} \cdot \frac{12}{13} = 0.213018.$$

So $\text{Var}[Y]$ is larger. Why does this make sense? Well, remember that variance measures spread. And by replacing cards after we draw them, we're increasing variability, or spread, of the remaining cards.