Indexed sets: union, intersection.

The idea: suppose we have a set A_α for each α in some set I.

<u>Example 1</u>: for each $\alpha \in \mathbb{R}$, define $A_\alpha = (-\alpha, \alpha) = \{x \in \mathbb{R} : -\alpha \le x \le \alpha\}$.

Then:

(a) We call each $\alpha \in I$ an <u>index</u>, or a <u>subscript</u>.

(b) We call I an <u>indexing set</u>.

E.g. in Example 1, the indexing set is $\mathbb{R}$; each real number α is an index.

(c) We define:

$$\bigcup_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for some } \alpha \in I\}$$

("the union of the A_α's over α in I"),

$$\bigcap_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for all } \alpha \in I\}$$

("the intersection of the A_α's over α in I").

E.g. for Example 1 above,

$$\bigcup_{\alpha \in \mathbb{R}} A_\alpha = \mathbb{R}, \qquad \bigcap_{\alpha \in \mathbb{R}} A_\alpha = \{0\}.$$

<u>Example 2.</u> Let $P = \{\text{positive prime numbers}\}$

$= \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots\}.$

Then

$$\bigcup_{p \in P} p\mathbb{Z} = \mathbb{Z} - \{\pm 1\},^{*} \quad \bigcap_{p \in P} p\mathbb{Z} = \{0\}.^{*}$$

*Every integer except +1 or -1 is divisible by some prime

* 0 is the only integer divisible by all primes (because there are infinitely many primes)

<u>Remark</u>. Suppose our indexing set I is a set of consecutive integers ranging from a to b (where a and/or b might be infinite). Then we write

$$\bigcup_{i=a}^{b} A_i \quad \text{for} \quad \bigcup_{i \in I} A_i,$$

and similarly for intersections.

<u>Example 3</u>.

$$\bigcup_{i=0}^{5} (i + 6\mathbb{Z}) = \mathbb{Z} \quad \text{and} \quad \bigcap_{i=0}^{5} (i + 6\mathbb{Z}) = \emptyset$$

(any integer a can be written $a = 6q + i$ where $q, i \in \mathbb{Z}$, $0 \leq i < 5$, and q, i are unique).

<u>Example 4.</u>

For each $i \in \mathbb{N}$, define

$$S_i = \left(\frac{1}{i+5}, \frac{1}{i} \right].$$

Then, for example,

$$\bigcup_{i=1}^{4} S_i = \left(\tfrac{1}{6}, 1\right] \cup \left(\tfrac{1}{7}, \tfrac{1}{2}\right] \cup \left(\tfrac{1}{8}, \tfrac{1}{3}\right] \cup \left(\tfrac{1}{9}, \tfrac{1}{4}\right]$$

$$= \left(\tfrac{1}{9}, 1\right],$$

$$\bigcap_{i=1}^{4} S_i = \left(\tfrac{1}{6}, \tfrac{1}{4}\right]$$ (think about it!),

$$\bigcup_{i=2}^{\infty} S_i = \left(0, \tfrac{1}{2}\right], \qquad \bigcap_{i=2}^{\infty} S_i = \emptyset.^{*}$$

*For example, $S_2 \cap S_8 = \left(\tfrac{1}{7}, \tfrac{1}{2}\right] \cap \left(\tfrac{1}{13}, \tfrac{1}{8}\right] = \emptyset$.

Example 5.

For each $r \in [0,1]$, define

$$C_r = [r,1] \times [0,r] = \{(x,y) \in \mathbb{R}^2 : r \le x \le 1,\ 0 \le y \le r\}.$$

Then

$$\bigcup_{r \in [0,1]} C_r = \{(x,y) \in [0,1] \times [0,1] : y \le x\}$$

and

$$\bigcap_{r \in [0,1]} C_r = \{(1,0)\}.$$

Picture: