More on sets.

Let A, B be sets.

Definitions 1-4.

We define:

1) The <u>union</u> $A \cup B$ ("A union B") by

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

(in math, "or" is inclusive: it means one or the other, or both).

2) The <u>intersection</u> $A \cap B$ ("A intersect B") by

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$
$$= \{x \in A : x \in B\} = \{x \in B : x \in A\}.$$

3) The <u>difference</u> $A - B$ ("A minus B", or "A complement B") by

$$A - B = \{x \in A : x \notin B\}.$$

4) The Cartesian product $A \times B$ ("A cross B") by

$$A \times B = \{\text{ordered pairs } (x,y) : x \in A,\ y \in B\}.$$

Example 1.

Let

$$A = (-30, 84],\quad B = [12, 157),\quad C = \{e, f, g\},$$

$$D = \{e, m\},\quad E = \{\{a,b\}, \{1,2,3\}, m, f, g\}.$$

Then:

$A \cup B = (-30, 157)$, $A \cap B = [12, 84]$,
$A - B = (-30, 12)$, $B - A = (84, 157)$,
$A \times B = \{(x,y) : -30 < x \leq 84,\ 12 \leq y < 157\}$
(a rectangle with part of its border missing).

$C \cup D = \{e, f, g, m\}$ $C \cap D = \{e\}$,
$C - D = \{f, g\}$, $D - C = \{m\}$,
$C \times D = \{(e,e), (e,m), (f,e), (f,m), (g,e), (g,m)\}$,

$D \cap E = \{m\}$, $E - C = \{\{a,b\}, \{1,2,3\}\}$, etc.

We can generalize Definitions 1, 2, 4 to more than two sets, e.g.

$C \cup D \cup E = \{e, f, g, m, \{a,b\}, \{1,2,3\}\}$,

$\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{\text{quadruples } (x,y,z,t) : x, y, z, t \in \mathbb{R}\}$

(also written $\mathbb{R}^4$), etc.

Def'n 3 is harder to generalize since, in general, $X - (Y - Z) \neq (X - Y) - Z$, for sets X, Y, Z. For example, consider $X = \{1,2,3,4,5,6\}$, $Y = \{1,2,3,4\}$, $Z = \{3,4,5,6\}$.

We can also mix operations, e.g. for C, D, E as above,

$C \cup (D \cap E) = C$, $(C - E) \times D = \{(e,e), (e,m)\}$, etc.

Remark: in general, we have $A \cup B = B \cup A$ and $A \cap B = B \cup A$, but $A - B \neq B - A$ and $A \times B \neq B \times A$.

Definition 5.

If S is a set, the power set $\mathcal{P}(S)$ of S is the set of all subsets of S.

Example.

For C, D as above,

$$\mathcal{P}(D) = \{ \emptyset, \{e\}, \{m\}, \{e,m\}\},$$

$$\mathcal{P}(C) = \{ \emptyset, \{e\}, \{f\}, \{g\}, \{e,f\}, \{e,g\}, \{f,g\}, \{e,f,g\}\}.$$

Note that $|\mathcal{P}(D)| = 4 = 2^{|D|}$, $|\mathcal{P}(C)| = 8 = 2^{|C|}$.

In general, we have

Theorem.

If S is a finite set ($|S| = n$ for some $n \in \mathbb{N}$), then

$$|\mathcal{P}(S)| = 2^{|S|}.$$

E.g. for the set E above,

$$|\mathcal{P}(E)| = 2^5 = 32.$$

Note: the theorem holds for infinite sets as well, if we define $2^\infty = \infty$.