Sets. (Chapter 1, BOP.)

Definition 1.

A set is a collection of distinct objects, called elements of the set.

Ways of describing/defining/denoting sets:
- words
- symbols/names
- listing elements
- "set builder notation".

Examples.

symbol/name

1) L = the set of distinct letters in the word "mathematikvergnugen" — words
$= \{a, e, g, h, i, k, m, n, r, t, u, v\}$
$= \{m, a, t, h, e, i, k, v, r, g, n, u\}$ — listings (order doesn't matter)
$= \{\text{letters } \alpha : \alpha \text{ is a letter in "mathematikvergnugen"}\}$ — set builder notation

Note: the braces mean "the set consisting of;" the colon means "such that."

2) $\mathbb{Z}$ = the set of all integers
$= \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$= \{0, \pm 1, \pm 2, \ldots\}$

The symbol $\mathbb{Z}$ is reserved for the set of integers.

3) E = the set of even integers $= \{0, \pm 2, \pm 4, \ldots\}$
$= \{2n : n \in \mathbb{Z}\}$

read "is an element of"

$= \{ n \in \mathbb{Z} : n = 2k \text{ for some } k \in \mathbb{Z} \}$.

4) $F = \{\text{even integers from } -6 \text{ to } 4 \text{ inclusive}\}$
$= \{ m \in E : -6 \leq m \leq 4 \} = \{-6, -4, -2, 0, 2, 4\}$

5) $2 + 7\mathbb{Z} = \{ n \in \mathbb{Z} : n = 2 + 7k \text{ for some } k \in \mathbb{Z} \}$
$= \{ \ldots, -12, -5, 2, 9, \ldots \}$

6) In general, for $a, b \in \mathbb{Z}$, $a + b\mathbb{Z}$ denotes $\{ n \in \mathbb{Z} : n = a + bk \text{ for some } k \in \mathbb{Z} \}$.
E.g. the set E above may be denoted $0 + 2\mathbb{Z}$, also written $2\mathbb{Z}$. Similarly, {odd integers} may be denoted $1 + 2\mathbb{Z}$.

7) $[-3, 5) = \{\text{real numbers } x : -3 \leq x < 5\}$.
$(-3, 5) = \{\text{real numbers } x : -3 < x < 5\}$.
Warning: $(-3, 5)$ also denotes a point in the plane!!

8) $SL(2, \mathbb{Z}) = \{\text{matrices } \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},$
$ad - bc = 1\}$.

9) More special, reserved symbols:

$\mathbb{R} = \{\text{real numbers}\}$
$\mathbb{R}^2 = \{\text{ordered pairs } (x, y) : x, y \in \mathbb{R}\}$
$\mathbb{N} = \{\text{natural numbers}\} = \{ n \in \mathbb{Z} : n > 0 \}$
$\mathbb{Q} = \{\text{rational numbers}\}$
$= \{ m/n : m, n \in \mathbb{Z} \text{ and } n \neq 0 \}$
$\emptyset$ = <u>the empty set</u> (the set with no elements), also denoted $\{ \}$.

10) You can have sets of sets, or sets containing sets and other things, like

$$\{\{1,2\},\{3\}\},\ \{\emptyset\},\ \{\{\pi, 5, \sqrt{2}\}, x, y, z\},$$
$$\{7, \{7\}, \{7,\{7\}\}\}.$$

Definition 2.

Let A, B be sets. We say <u>A is a subset of B</u>, written $A \subseteq B$, if every element of A is also in B (that is: if no element of A lies outside of B). Otherwise, we write $A \nsubseteq B$.

E.g. for the sets defined above:

$\mathbb{N} \subseteq \mathbb{Z}$; $\mathbb{Z} \subseteq \mathbb{R}$ (we can write $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{R}$);
$2+7\mathbb{Z} \subseteq \mathbb{Z}$; $[-3,5) \subseteq \mathbb{R}$; $F \subseteq E$; $\mathbb{Q} \nsubseteq \mathbb{Z}$; $F \nsubseteq 2+7\mathbb{Z}$;

$\{\{1,2\}\} \subseteq \{\{1,2\},\{3\}\}$,
$\emptyset \subseteq$ any set whatsoever;
any set whatsoever $\subseteq$ itself.

Definition 3

The <u>cardinality</u> of a set S, denoted $|S|$, is the number of elements of S.

E.g. for the sets above,

$|F| = 6$, $|L| = 12$, $|\emptyset| = 0$,

$$|\{\{1,2\},\{3\}\}| = 2,$$

$$|\mathbb{N}| = |\mathbb{Q}| = |\mathbb{R}| = |[-3,5)| = |E| = |2+7\mathbb{Z}| = |SL(2,\mathbb{Z})|$$
$$= \infty.$$