Euclidean algorithm concluded:

Q: what to do if the algorithm gives you

$$ax - by = \gcd(a,b) \qquad (*)$$

where x, y are not positive?

A: add any multiple of b — say nb, where $n \in \mathbb{N}$ — to x, and add the same multiple na of a to y.

You get

$$a(x+nb) - b(y+na) = \gcd(a,b)$$

which is true whenever $(*)$ is.

Example
Let $k = 31$ and $m = 55$.
Show that $\gcd(k, \varphi(m)) = 1$ and find $x, y \in \mathbb{N}$ with

$$kx - \varphi(m)y = 1.$$

Solution.

$$\varphi(55) = \varphi(5 \cdot 11) = 4 \cdot 10 = 40.$$

We have

$$40 = 31 \cdot 1 + 9$$
$$31 = 9 \cdot 3 + 4$$
$$9 = 4 \cdot 2 + 1$$
$$4 = 4 \cdot 1 + 0$$

So $\gcd(k, \varphi(m) = 1)$. Also:

$$
\begin{aligned}
1 &= 9-4\cdot 2 \\
&= 9-(31-9\cdot 3)\cdot 2 \\
&= 9\cdot 7-31\cdot 2 \\
&= (40-31\cdot 1)\cdot 7-31\cdot 2 \\
&= 40\cdot 7-31\cdot 9.
\end{aligned}
$$

That is,

$$31(-9)-40(-7)=1.$$

So we add 40 to -9 and 31 to -7:

$$31(-9+40)-40(-7+31)=\underline{1}$$

$$31\cdot 31-40\cdot 24=1.$$

(DIY: check this.)