In this activity, we explore the **Inclusion-Exclusion Principle (IEP)** and the **Multiplication Principle (MP)**, which help us count the elements in unions of sets.

Recall: If S is a set, then $|S|$, called the *cardinality* of S, denotes the number of elements of S. Throughout this activity, we'll assume that every set S is *finite*, meaning $|S|$ is a finite number.

1. **The IEP for two sets.** Fill in the blanks; each blank should involve only the symbols A, B, $\cup$, $\cap$, and $|\ |$ (that is, the vertical bars used to denote the cardinality of a set), perhaps in combination with each other.

   Let A and B be sets. Then the set of elements in A or B (or both) is the set ___$A \cup B$___. To count the elements in this set, we count the number of elements of A, add to this the number of elements of ___B___, and then, *to compensate for overcounting* (since we've counted the elements that are in both ___A___ and B twice), we subtract the number of elements in ___$A \cap B$___. In other words, we have the formula

   $$|A \cup B| = |A| + \underline{\quad |B| \quad} - \underline{\quad |A \cap B| \quad}.$$

2. **The IEP for three sets.** Let's generalize the above to three sets. Fill in the blanks with the symbols A, B, C, $\cup$, $\cap$, and $|\ |$, perhaps in combination with each other.

   Let A, B, and C be sets. Then the set of items in A or B or C (with the word "or" being used in the inclusive sense, as usual) is ___$A \cup B \cup C$___. To count the elements in this set, we count the number of elements of A, add to this the number of elements of ___B___, and add to this the number of elements of ___C___. But we've overcompensated: we've counted elements of $A \cap B$, and of ___$A \cap C$___, and of ___$B \cap C$___, twice. To compensate, we subtract the number of elements in each of these sets.

   But wait, we're not done! By adding $|A|$ to $|B|$ to ___$|C|$___, we've counted the elements in $A \cap B \cap C$ ___three___ times. Moreover, in subtracting $|A \cap B|$ and ___$|A \cap C|$___ and ___$|B \cap C|$___, we've subtracted off the elements in $A \cap B \cap C$ ___three___ times. So the net effect is to count the elements in $A \cap B \cap C$ ___zero___ times. But we don't want to count them that many times, we want to count them exactly ___once___. So let's put them back!

   In other words, all told, we have the formula

   $$\begin{aligned} |A \cup B \cup C| = |A| &+ \underline{\quad |B| \quad} + \underline{\quad |C| \quad} \\ &- |A \cap B| - \underline{\quad |A \cap C| \quad} - \underline{\quad |B \cap C| \quad} \\ &+ \underline{\quad |A \cap B \cap C| \quad}. \end{aligned}$$

3. **Counting with the IEP and the MP.** For this problem, we use not only the result of Exercise 2 above, but also the **Multiplication Principle (MP)**, which says: if there are m ways of doing Thing 1, and for each of these ways, there are n ways of doing Thing 2, then the number of ways of doing Thing 1 followed by Thing 2 is mn.

   Using these ideas, solve the following problems. For simplicity, assume that all letters are lower-case English letters, and that y is a consonant (not a vowel). Also, throughout, it's OK to repeat letters.

   (a) Find the number of two-letter strings (i.e. "words," though they don't have to be actual words that you'd find in a dictionary) that start with a vowel. Use MP.

   We'll do this one for you, so you get the idea:

   There are five vowels (a,e,i,o,u), so there are five choices for the first letter. There are 26 letters, so 26 choices for the second letter. So by MP, there are

$$5 \cdot 26 = 130$$

   strings of the given form.

   (b) Find the number of two-letter strings that end with a vowel. Use MP.

   $26 \cdot 5 = 130.$

   (c) Find the number of two-letter strings that start *and* end with a vowel (the starting and ending vowels could be the same or different). Use MP.

   $5 \cdot 5 = 25.$

(d) Find the number of two-letter strings where the two letters are different. Use MP.

$26 \cdot 25 = 650.$

(e) Find the number of two-letter strings that start with a vowel, *and* where the two letters are different. Use MP.

$5 \cdot 25 = 125.$

(f) Find the number of two-letter strings that end with a vowel, *and* where the two letters are different. Use MP.

$25 \cdot 5 = 125.$

(g) Find the number of two-letter strings that start with a vowel, *and* end with a vowel, *and* where the two letters are different. Use MP.
$5 \cdot 4 = 20.$

(h) Find the number of two-letter strings that start with a vowel, *or* end with a vowel, *or* where the two letters are different. Use the IEP for three sets (above; Exercise 2).
$130 + 130 + 650 - 25 - 125 - 125 + 20 = 655.$