Stuff about Proofs and Other Phenomena (S-POP)

Part A: Communicating mathematics.

Math is hard (according to reliable sources). But there are things you can do to make it easier. One of the main things is the following:

> **BIG IDEA.** You can make math easier, for your audience and **yourself**, by making the effort to *communicate* mathematics clearly.

Note the word "clearly." What does that mean? We'll get to that shortly. In the meantime, there's another important word in the above **BIG IDEA**—the word "yourself." That word is there to highlight:

> **ANOTHER BIG IDEA.** Clarity of communication not only **reflects**, but also **reinforces**, clarity of thought.

That is: not only do you need to understand it well to explain it well, but if you can explain it well, then most likely you've understood it well too.

Back to communicating mathematics clearly, then. Here are some tips. Please note that a number of these are adapted from Section 5.3 of *Book of Proof* by Richard Hammack. (There's a link to this text on our Canvas page.) We've not included all of those tips because we don't necessarily subscribe to all of them. As Hammack himself writes,

> Unlike logic and mathematics, where there is a clear-cut distinction between what is right or wrong, the difference between good and bad writing is sometimes a matter of opinion.

In summary, use your judgement, but you may use the following to guide you.

1. **Speak in complete sentences.** The nice thing about math (well, one nice thing, anyway; there are many others) is that complete sentences can be quite brief: for example, "$x \geq 2$" is a complete sentence. It has a subject, an object, and a verb; what else do you need?

2. **Use math symbols appropriately.** In particular, don't use math symbols like "$\Rightarrow$" or "$\rightarrow$" to mean "equals." The symbol "$\Rightarrow$" means "implies" (example: S is a square $\Rightarrow$ S has four right angles; see Section B(i) below); the symbol "$\rightarrow$" means "converges to" or "approaches" (example: $\lim_{x\to 0}(\sin x)/x = 1$; see Section B(iv) below). Please do not use either of these symbols to denote equality.

3. **Separate mathematical expressions and statements with "connective tissue" (words).** For example, consider the statement

$$\text{Since } 2x + 4 \geq 0,\ x \geq -2.$$

The meaning is not so clear; it's better to write

Since $2x + 4 \geq 0$, it follows that $x \geq -2$

or

Since $2x + 4 \geq 0$, we see that $x \geq -2$.

4. **Write in the first person plural (sometimes called "the editorial 'we' ").** For example, write "We will now show that..." instead of "I will now show that...." It's more inviting, and it's completely standard in mathematics writing.

5. **Explain each new symbol.** For example, instead of "If n is an even integer, then $n = 2k$," it would be better to write "If n is an even integer, then $n = 2k$ for some integer k." Otherwise, the reader might flip back though pages trying to figure out what k is and where it was defined, when in fact it has not been.

6. **Be specific.** Avoid words like "it" or "the expression" or "the given quantity" if these words are ambiguous. For example, instead of "If m is an integer, $k = 2m + 1$, and $n = 2k + 1$, then it's odd," or "If m is an integer, $k = 2m + 1$, and $n = 2k + 1$, then the given quantity odd," write "If m is an integer, $k = 2m + 1$, and $n = 2k + 1$, then n is odd." (After all, "If m is an integer, $k = 2m + 1$, and $n = 2k + 1$, then k is odd" is also mathematically correct.)

Part B: Some varieties of mathematical proof

What follows is a brief, and by no means exhaustive (though perhaps a bit tiring), introduction to various varieties and strategies of mathematical argument and proof.

Section B(i): $P \Rightarrow Q$ and related results.

If P and Q are statements of any kind, then the statement "$P \Rightarrow Q$," read "P implies Q," means anytime P is true, Q follows. Other ways of saying "$P \Rightarrow Q$" are: "if P, then Q;" "Q if P;" "Q whenever P;" "Q if P;" "P only if Q."

To be more precise about what "$P \Rightarrow Q$" means, from a formal point of view, we'd need to get into truth tables and so on. If you're interested, see Section 2 of *Book of Proof*. Instead, we take it on faith that we understand a statement like "today is Saturday implies it's the weekend." (Convince yourself that this statement has the same meaning as "if today is Saturday, then it's the weekend," "it's the weekend if today is Saturday," "today is Saturday only if it's the weekend.")

The so-called *direct proof* of a statement like $P \Rightarrow Q$ goes as follows (we will discuss other methods of proof a bit later):

Proposition B(i)-1. $P \Rightarrow Q$.

Proof. Assume P. [Anything you see in square-brackets is intended not as part of the proof in question, but as a note about what's going on. In this case, what's going on is that you have to do some stuff here to get to the point where you can conclude:] Therefore, Q.

So $P \Rightarrow Q$. $\square$

Remark A. You should always end a proof with some kind of clear indication that you are done. Here we have used a box ("$\square$"); you'll see this notation frequently in mathematical texts and research articles. Alternatively, you might want to use "QED," or some other indicator.

Remark B. The last line ("So $P \Rightarrow Q$") of the above proof summarizes what has been done. A line like this is not, strictly speaking, necessary, but does serve as a useful reminder. Such a reminder is especially useful in longer proofs, where there's so much "stuff" in between the P and the Q that the upshot of the whole argument bears emphasizing. In shorter proofs, you might find that such an concluding line is unnecessary. Use your judgement.

To illustrate Proposition B(i)-1, let's recall that an *even number* is an integer that equals $2k$ for some integer k; an *odd number* is an integer that equals $2k-1$ for some integer k. We have:

Proposition B(i)-1$_{\mathbf{E}}$. If n is an even number then $n-1$ is an odd number.

Proof. Assume n is an even number. We may write $n = 2k$ for some integer k. But then $n-1 = 2k-1$, so $n-1$ is odd.

So if n is an even number, then $n-1$ is an odd number. $\square$

(The "E" in the subscript of Proposition B(i)-1$_{\mathrm{E}}$ indicates that this proposition *exemplifies* Proposition B(i)-1.)

Remark. It may be tempting to omit the first line ("Assume n is an even number") of the above proof. We recommend against it. In a direct proof of $P \Rightarrow Q$, we are not simply proving that Q is true; we're proving that Q follows from the assumption of P. That assumption is perhaps *implicit* in the statement of the proposition, but it's best to make it *explicit* in the proof.

Exercise B(i)-1. (a) Prove that the sum of two odd numbers is even. **(b)** Prove that the product of two odd numbers is odd.

Exercise B(i)-2. (a) Prove that, if n is an even number, then n^2 is divisible by 4. **(b)** Prove that, if n is an odd number, then $n^2 - 1$ is divisible by 4.

Exercise B(i)-3. Let a, b, and c be integers. Recall that we say "a divides b," written $a|b$, if there exists an integer q such that $b = aq$. **(a)** Prove that, if $a|b$ and $a|c$, then $a|(b+c)$. **(b)** Prove that, if $a|b$, then $a|nb$ for any integer n.

We next note that the statement $P \Rightarrow Q$ is (always, always, ALWAYS) logically equivalent to its *contrapositive:* the latter is, by definition, the statement $\sim Q \Rightarrow \sim P$. Here, the symbol "$\sim$" stands for "not:" so $\sim P$ means "not P," or "the negation of P." (Sometimes you'll see "$\neg$" used in place of "$\sim$.") Think about it: for example, the contrapositive of "if today is Saturday, then it's the weekend" is "if it's not the weekend, then today is not Saturday." The two statements mean the same thing.

We can therefore also prove $P \Rightarrow Q$ by *contraposition*, as follows:

Proposition B(i)-2. $P \Rightarrow Q$.

Proof. Assume $\sim Q$. [Do what you got to, to get to:] Therefore, $\sim P$.
So $P \Rightarrow Q$. □

For example:

Proposition B(i)-2$_{\mathbf{E}}$. If m^2 is an odd number, then m is an odd number.

Proof. Suppose m is not odd. Then m is even, so $m = 2k$ for some k. But then $m^2 = (2k)^2 = 4k^2 = 2(2k^2)$, so m^2 is even, and hence not odd.
So if m^2 is odd, then m is odd. □

Remark. In the above proof, we used the fact that every integer m must be either even or odd, but not both. This may seem obvious, but is worth explaining. It follows from the *division algorithm*, which tells us we can divide 2 into m, to get a unique integer quotient k and a unique non-negative integer remainder r, and this remainder must be less than the divisor 2. That is, $m = 2k + r$ where k and r are integers, and either r equals 0 (in which case m is even) or 1 (in which case m is odd).

More generally, the **division algorithm** tells us that we can alway divide an integer by another, positive integer, to get a quotient and a remainder that's smaller than the divisor. More formally:

The division algorithm. Given an integer m and a positive integer b, there are unique integers q and r with $m = bq + r$ and $0 \leq r < b$.

The division algorithm also probably seems pretty obvious, or at least familiar. But in fact, if follows from a fairly deep fact, called the *well-ordering principle*, concerning the integers. See, for example, Axiom 3.1.1, p. 105, of our course text. We won't discuss this further, except to say that what's obvious in mathematics is sometimes quite profound. (In fact, often, the more obvious, the more profound.)

Exercise B(i)-4. Supply a proof by contraposition of Proposition B(i)-1$_{\mathrm{E}}$.

Exercise B(i)-5. Supply a direct proof of Proposition B(i)-2$_{\mathrm{E}}$. Hint: Suppose m^2 is odd. Then we can write $m^2 = 2\ell + 1$ where ℓ is an integer. Now write $m = 2k + r$, where k is an integer, and r equals either 0 or 1. (Why can we write m this way?) Now we have two different ways of writing m^2; set them equal, do some algebra, and see what you can deduce about r.

Exercise B(i)-6. Using contraposition prove that, if n is not divisible by 4, then n is not divisible by 12.

Whether, in a given instance, to use the direct or the contraposition method to prove $P \Rightarrow Q$ comes down to a matter of choice; which one seems to work better in the given situation?

WARNING: $P \Rightarrow Q$ is *not* logically equivalent to its *converse,* meaning the statement $Q \Rightarrow P$. For example, "If today is Saturday then it's the weekend" is not equivalent to

"If it's the weekend, then today is Saturday." (After all, if it's the weekend, it might be Sunday.) So don't ever try to prove $P \Rightarrow Q$ by assuming Q, and deducing P.

Incidentally, in the above paragraph, we have demonstrated that the statement "If it's the weekend, then today is Saturday" is false by the method of *counterexample.* That is, we have produced a *single* scenario where the statement doesn't hold. In general, to prove that a statement $X \Rightarrow Y$ is false, it's enough to exhibit a single situation where X holds but Y doesn't.

Exercise B(i)-7. Consider the statement:

If x is odd, then x is divisible by 3.

Prove that this statement is false, using the method of counterexample.

Exercise B(i)-8. Consider the converse to the statement of Exercise B(i)-3(a). Is this converse statement true? If so, prove it. If not, show that it's false by counterexample.

Finally, we note that the statement $P \Leftrightarrow Q$, read "P if and only if Q," or more briefly "P iff Q," by definition means $P \Rightarrow Q$ *and* $Q \Rightarrow P$. One way to prove $P \Leftrightarrow Q$ is to demonstrate one at a time the two statements it comprises – that is:

Proposition B(i)-3. $P \Leftrightarrow Q$.

Proof. (a) First we show that $P \Rightarrow Q$: assume P. [Do some stuff in here to get to:] Therefore, Q.

So $P \Rightarrow Q$, as required.

(b) Now we show that $Q \Rightarrow P$: assume Q. [Do some stuff in here to get to:] Therefore, P.

So $Q \Rightarrow P$, as required.

Since $P \Rightarrow Q$ and $Q \Rightarrow P$, we conclude that $P \Leftrightarrow Q$. $\square$

Exercise B(i)-9. Use the method outlined in Proposition B(i)-3 to show that an integer n is divisible by 6 if, and only if, n is both even and divisible by 3. Hint for one of the directions: note that, if n is divisible by 3, then $n = 3k$ for some integer k. Now if n is also divisible by 2 – that is, if n is even – what does the equation $n = 3k$ tell you about k? Use Exercise B(i)-1(b).

Remark. An "if and only if" proof can sometimes be shortened by observing that each step in the proof not only is implied by, but also implies, the previous one. For example, consider the following:

Proposition B(i)-3$_{\mathbf{E}}$. n is an even number if and only if $n - 1$ is an odd number.

Proof. n is even iff $n = 2k$ for some integer k, which is true iff $n - 1 = 2k - 1$ for some integer k, which is true iff $n - 1$ is odd.

So n is even iff $n - 1$ is odd. $\square$

Exercise B(i)-10. Let x be a real number. Use the method of proof shown in Proposition B(i)-3$_{\mathrm{E}}$ to show that $x^2 = 1$ iff $x = -1$ or $x = 1$. (Pretend you didn't know this already.)

Hint: $x^2 = 1$ iff $x^2 - 1$=0. Now factor $x^2 - 1$, and use the fact that, for a, b real numbers, the product ab equals zero iff $a = 0$ or $b = 0$ (or both).

Part B(ii): $A \subseteq B$ and related results

As a nice illustration of what can be done with the idea of $P \Rightarrow Q$, we consider the statement $A \subseteq B$. Here A and B are sets; the statement $A \subseteq B$ is read "A is a subset of B," which just means A is *contained* in B, which just means every element of A is also an element of B, which just means $P \Rightarrow Q$, where P is the statement "$x \in A$" and Q the statement "$x \in B$." (The symbol "$\in$" is read "is an element of.") So an "$A \subseteq B$" result *is* a "$P \Rightarrow Q$" result, of a certain kind.

Proposition B(ii)-1. $A \subseteq B$.

Proof. Let $x \in A$. [Now do what you've got to, to get to:] Therefore, $x \in B$.

So $A \subseteq B$. □

As an easy, but illustrative, example, let's recall that, for general sets S and T, $S \cap T$ means the set of all objects that belong to S *and* belong to T. Then:

Proposition B(ii)-$1_{\mathbf{E}}$. For any sets S and T, we have $S \cap T \subseteq S$.

Proof. Let $x \in S \cap T$. Then $x \in S$ and $x \in T$, so in particular, $x \in S$. So $S \cap T \subseteq S$. □

Exercise B(ii)-1. Show that the set of all integer multiples of 4 is contained in the set of all even numbers.

Exercise B(ii)-2. Show that the set of all perfect fourth powers is contained in the set of all perfect squares. (A perfect fourth power in an integer m such that $m = \ell^4$ for some integer ℓ; similarly we define perfect squares.)

For the next two exercises, recall that, for sets S and T, $S \cup T$ denotes the union of S and T, meaning the set of all things in S or in T.

Remark. In mathematics, the word "or" is always, unless otherwise specified, used in the *inclusive* sense. That is, a mathematical statement of the form "X or Y" will always, unless otherwise stated, mean "X or Y or *both*." In particular, $S \cup T$ denotes the set of objects either in S, or in T, or perhaps in both. For example, $\{$integer multiples of 3$\} \cup \{$integer multiples of 5$\}$ *includes* the number 45, since 45 is both a multiple of 3 *and* a multiple of 5.

Exercise B(ii)-3. Show that, for any sets S and T, we have $S \subseteq S \cup T$.

Exercise B(ii)-4. Show that, for any sets A, B, and C, we have $A \cap B \subseteq A \cup C$.

Now two sets are, by definition, equal if each is contained in the other: so to *prove* two sets A and B are equal, it's enough to prove that $A \subseteq B$ and that $B \subseteq A$. Like this:

Proposition B(ii)-2. $A = B$.

Proof. (a) We first show that $A \subseteq B$: let $x \in A$. [Now go until you get to:] Therefore, $x \in B$. So $A \subseteq B$, as required.

(b) We now show $B \subseteq A$: let $x \in B$. [Now go until you get to:] Therefore, $x \in A$. So $B \subseteq A$, as required.

Since $A \subseteq B$ and $B \subseteq A$, we have $A = B$. $\square$

As a concrete example, let's define the *symmetric difference* $C \Delta D$ of sets C and D by

$$C \Delta D \overset{\text{def'n}}{=} (C - D) \cup (D - C).$$

(Recall: in general $A - B$ means the set of all things in A but not in B.) We have:

Proposition B(ii)-2$_\mathbf{E}$. $C \Delta D = (C \cup D) - (C \cap D)$.

Proof. (a) We show $C \Delta D \subseteq (C \cup D) - (C \cap D)$: let $x \in C \Delta D$. By definition of $C \Delta D$, this means $x \in C - D$ or $x \in D - C$. If $x \in C - D$ then $x \in C$, so $x \in C \cup D$, but $x \notin D$, so $x \notin C \cap D$. Therefore $x \in (C \cup D) - (C \cap D)$. On the other hand, if $x \in D - C$ then $x \in D$, so $x \in C \cup D$, but $x \notin C$, so $x \notin C \cap D$. Therefore $x \in (C \cup D) - (C \cap D)$. So in either case $x \in (C \cup D) - (C \cap D)$. So $C \Delta D \subseteq (C \cup D) - (C \cap D)$, as required.

(b) We show $(C \cup D) - (C \cap D) \subseteq C \Delta D$: let $x \in (C \cup D) - (C \cap D)$. Then $x \in C \cup D$ but $x \notin C \cap D$. Now since $x \in C \cup D$, we know $x \in C$ or $x \in D$. If $x \in C$ then, since $x \notin C \cap D$, we have $x \notin D$, so $x \in C - D$, whence $x \in C \Delta D$ (by definition of $C \Delta D$). If $x \in D$ then, since $x \notin C \cap D$, we have $x \notin C$, so $x \in D - C$, whence $x \in C \Delta D$ (by definition of $C \Delta D$). So in either case $x \in C \Delta D$. So $(C \cup D) - (C \cap D) \subseteq C \Delta D$, as required.

Since $C \Delta D \subseteq (C \cup D) - (C \cap D)$ and $(C \cup D) - (C \cap D) \subseteq C \Delta D$, we have $C \Delta D = (C \cup D) - (C \cap D)$. $\square$

(Suggestion: draw a Venn diagram to understand symmetric differences and Proposition B(i)-5$_\mathrm{E}$.)

Exercise B(ii)-5. Let $\mathbb{Z}$ denote the set of integers. Also, for given integers b and r, let $r + b\mathbb{Z}$ denotes the set of all integers of the form $r + bq$ for some integer q. (For example, $3 + 7\mathbb{Z} = \{\ldots, 3 + 7(-3), 3 + 7(-2), 3 + 7(-1), 3 + 7(0), 3 + 7(1), 3 + 7(2), 3 + 7(3), \ldots\} = \{\ldots, -18, -11, -4, 3, 10, 17, 24, \ldots\}$.)

Using the strategy of Proposition B(ii)-2, prove that

$$\mathbb{Z} = 3\mathbb{Z} \cup (1 + 3\mathbb{Z}) \cup (2 + 3\mathbb{Z}),$$

where $3\mathbb{Z}$ is shorthand for $0 + 3\mathbb{Z}$, and in general, $A \cup B \cup C$ denotes the set of all objects either in A, or B, or C.

Hint: use the division algorithm, referenced at the top of p. 4 above, and on p. 29 of T-BOP.

Exercise B(ii)-6. Is the union $3\mathbb{Z} \cup (1 + 3\mathbb{Z}) \cup (2 + 3\mathbb{Z})$ described in the previous problem *disjoint*? That is, do any two of the sets $3\mathbb{Z}$, $1 + 3\mathbb{Z}$, and $2 + 3\mathbb{Z}$ have any elements in common? Hint: think about the *uniqueness* described in the division algorithm.

Exercise B(ii)-7. Using the strategy of Proposition B(ii)-2, prove that, for any sets X, Y, and Z,

$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z).$$

(It may help to draw a Venn diagram to help you understand what's going on here. But a Venn diagram does not suffice here for a *proof.*)

Part B(iii): $\forall x \in X,\ Q(x)$; $\exists x \in X : Q(x)$, and related results

If $Q(x)$ is a statement regarding a generic element x of a set X, then the statement "$\forall x \in X,\ Q(x)$" means "for all $x \in X$, $Q(x)$ is true." Thus the "$\forall$," called the *universal quantifier,* means "for all."

The statements "$\forall x \in X,\ Q(x)$," "for any $x \in X$, $Q(x)$ is true," "given $x \in X$, $Q(x)$ is true," and "if $x \in X$, then $Q(x)$ is true" all mean the same thing. In particular, the last statement is of the form $P \Rightarrow Q$. So, recalling part **B(i)** above, we have the following strategy for proving "$\forall x \in X,\ Q(x)$:"

Proposition B(iii)-1. $\forall x \in X,\ Q(x)$.

Proof. Assume $x \in X$. [Now do what you got to, to get to:] Therefore, $Q(x)$.
So $\forall x \in X,\ Q(x)$. $\square$

For example:

Proposition B(iii)-1$_{\mathbf{E}}$. $\forall p \in \{\text{prime numbers}\} - \{3\}, 3$ divides $p^2 + 2$. [That is: if p is any prime number not equal to 3, then 3 divides $p^2 + 2$.]

Proof. Assume p is prime and not equal to 3. Then p is not divisible by 3 (a prime is only divisible by itself and 1), so if we divide 3 into p, we geta quotient k and a *nonzero* remainder r. Since r must be < 3, we have $r = 1$ or $r = 2$. That is, $p = 3k + r$ for some integer k, and $r = 1$ or $r = 2$.

But note, then, that

$$p^2 + 2 = (3k + r)^2 + 2 = 9k^2 + 6kr + r^2 + 2 = 3(3k^2 + 2kr) + r^2 + 2. \qquad (\circ)$$

If $r = 1$ then $r^2 + 2 = 3$; if $r = 2$ then $r^2 + 2 = 6$; in either case $r^2 + 2$ is a multiple of 3. That is, in either case $r^2 + 2 = 3m$ for some m. So $(\circ)$ gives

$$p^2 + 2 = 3(3k^2 + 2kr + m),$$

which implies that $p^2 + 2$ is a multiple of 3.
Therefore, $\forall p \in \{\text{prime numbers}\} - \{3\}, 3$ divides $p^2 + 2$. $\square$

Exercise B(iii)-1. Prove that, if m is an integer, then product $m(m+1)(m+2)$ is divisible by 6. In other words prove that, $\forall m \in \mathbb{Z},\ 6|m(m+1)(m+2)$.

Exercise B(iii)-2. Prove that, $\forall x, y \in \mathbb{R}$ (recall that $\mathbb{R}$ denotes the set of real numbers), we have

$$x^2 + y^2 \geq 6x + 4y - 15.$$

Hint: complete the squares. (Note: "$\forall x, y \in \mathbb{R}$" means "if $x \in \mathbb{R}$ *and* $y \in \mathbb{R}$.")

We next consider the sentence "$\exists x \in X : Q(x)$, " which means "for some $x \in X$, $Q(x)$ is true." Thus the "$\exists$," called the *existential quantifier,* means "for some."

The statements "$\exists x \in X : Q(x)$," "there is an $x \in X$ such that $Q(x)$ is true," and "there's at least one $x \in X$ such that $Q(x)$ is true" all mean the same thing. The most direct method of proving a statement like $\exists x \in X : Q(x)$ is by *finding* an $x \in X$ such that $Q(x)$ holds. Such a proof is called *constructive*, and looks like this:

Proposition B(iii)-2. $\exists x \in X : Q(x)$.

Proof. Let $x =$ [some element of X you've found that such that $Q(x)$ holds. Then *show* that $Q(x)$ holds, for this x, to conclude:] Then $Q(x)$ holds.

So $\exists x \in X : Q(x)$. □

Remark on constructive proofs: generally, you *do not* need to show the work that went into *finding* the x that works. However, once you have produced this x, you do *show* that it works (that is, it makes $Q(x)$ true).

Proposition B(iii)-2$_{\mathbf{E}}$. $\exists k \in \{\text{integers between 30 and 50}\} : k$ divides 576.

Proof. Let $k = 48$. Then $576 = 12k$, so k divides 576.

So $\exists k \in \{\text{numbers between 30 and 50}\} : k$ divides 576. □

(The "work" that goes into finding the number k appropriate for the above proposition amounts simply to checking all numbers between 30 and 50 'til one does the job. Again, in the proof you don't show this work; you just present the result, and show that it *does* do the job.)

Exercise B(iii)-3. Prove that $\exists p \in \{\text{prime numbers}\}$ such that such that $p > 100$.

Exercise B(iii)-4. Prove that $\exists k \in \mathbb{Z}$ such that k can be expressed as a sum of two squares in two different ways. Hint: you don't have to go too far; there's a $k < 100$ that works.

Quantifiers can be combined in various ways; for example, we can form statements like "$\forall x \in X, \exists y \in Y : Q(x, y)$." We'll consider a particularly useful context for such a combination in the next section. In the meantime, we note that great care should be taken with such combinations. In particular, the *order* of combination *matters*, as the following exercise attests.

Exercise B(iii)-5. (a) Prove that:

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \colon x > y.$$

(b) Prove that the statement

$$\exists y \in \mathbb{R} \colon \forall x \in \mathbb{R}, x > y$$

is false.

Part B(iv): Quantifiers and limits

One area where quantifiers may be applied quite nicely is in the discussion of *limits*. Specifically: let

$$x_1, x_2, x_3, \ldots$$

be a sequence of real numbers. Recall: to say

$$\lim_{n\to\infty} x_n = L \qquad (*)$$

is, intuitively, to say that x_n gets closer and closer to L as n gets larger and larger. Or, somewhat more precisely: $(*)$ means we can make x_n as close as we want to L, by making n large enough. Or, even more precisely: $(*)$ means we can make $|x_n - L|$ as small as we want, by making n large enough. Or, still *more* precisely, it means we can make $|x_n - L|$ smaller than any prescribed positive tolerance, call it ε, by making n large enough—say, larger than some specified real number R.

In other (more mathematical) words, $(*)$ means: *for any $\varepsilon > 0$, there exists an $R \in \mathbb{R}$ such that, if $n > R$, then $|x_n - L| < \varepsilon$.*

So, in light of what we've discussed above concerning the phrases "for any," "there exists," and "if P then Q," we are ready for:

Definition B(iv)-1. We say

$$\lim_{n\to\infty} x_n = L$$

if, $\forall\varepsilon > 0, \exists R \in \mathbb{R}$ such that $n > R \Rightarrow |x_n - L| < \varepsilon$.

So that's the definition. (It's due to Cauchy, ca. 1827.) It does require some practice to get one's brain around this definition, to the point of being able to *use* it to prove things.

To this end, let's begin with a template for a limit proof.

Proposition B(iv)-1.

$$\lim_{n\to\infty} x_n = L.$$

Proof. Let $\varepsilon > 0$. [Now, perform some algebra "scratchwork" (which you won't show as part of your proof) on the condition $|x_n - L| < \varepsilon$, to determine how large n has to be to make this condition true. Let's say you find that $|x_n - L| < \varepsilon$ whenever $n > R$, where R is some real number. Then $n > R \Rightarrow |x_n - L| < \varepsilon$. Then the following is what you write.] Let $R =$(whatever R works, according to your scratchwork). Then $n > R \Rightarrow$ (here you do algebra that's essentially the reverse of what you did to *find* R, to show that) $|x_n - L| < \varepsilon$. So $\lim_{n\to\infty} x_n = L$. $\square$

Now let's work a couple of examples, to gain familiarity with the relevant ideas.

Proposition B(iv)-1$_\mathbf{E}$.

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Proof. Let $\varepsilon > 0$. [Here's our scratch work: we want n large enough that $|1/n - 0| < \varepsilon$. But $|1/n - 0| = 1/n$ (since n is a positive integer). So we want $1/n < \varepsilon$, or $n > 1/\varepsilon$. So we choose $R = 1/\varepsilon$. Now this is what we write.] Let $R = 1/\varepsilon$. If $n > R$, then

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \frac{1}{R} = \frac{1}{1/\varepsilon} = \varepsilon.$$

So by Definition B(iv)-1,

$$\lim_{n\to\infty} \frac{1}{n} = 0,$$

as required. □

Exercise B(iv)-1. Show that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

Use only arguments involving ε and R, as in the proof of Proposition B(iv)-1_{E} above. That is, you're not allowed to use limit laws like "the limit of the square roots is the square root of the limits" or what have you.

Here's another example.

Proposition B(iv)-$1_{\mathbf{EE}}$.

$$\lim_{n\to\infty} \frac{2n + (-1)^n}{n+1} = 2.$$

Proof. Let $\varepsilon > 0$. [Scratchwork: We want n to be large enough that

$$\left|\frac{2n + (-1)^n}{n+1} - 2\right| < \varepsilon.$$

But note that

$$\left|\frac{2n + (-1)^n}{n+1} - 2\right| = \left|\frac{2n + (-1)^n - 2(n+1)}{n+1}\right| = \left|\frac{(-1)^n - 2}{n+1}\right| \le \frac{1+2}{n+1} = \frac{3}{n+1} < \frac{3}{n}.$$

(We used the triangle inequality on $\mathbb{R}$, which says that

$$|x + y| \le |x| + |y| \quad \forall x, y \in \mathbb{R}, \tag{$\triangle_=$}$$

to get $|(-1)^n - 2| \le |(-1)^n| + |-2| = 1 + 2 = 3$.) If we can make $3/n < \varepsilon$ then we'll be done. Solving $3/n < \varepsilon$ for n gives $n > 3/\varepsilon$. OK, now here's what we write.] Let $\varepsilon > 0$ be given. Let $R = 3/\varepsilon$. If $n > R$, then

$$\begin{aligned}\left|\frac{2n + (-1)^n}{n+1} - 2\right| &= \left|\frac{2n + (-1)^n - 2(n+1)}{n+1}\right| = \left|\frac{(-1)^n - 2}{n+1}\right| \le \frac{1+2}{n+1} = \frac{3}{n+1} \\ &< \frac{3}{n} < \frac{3}{R} = \frac{3}{3/\varepsilon} = \varepsilon.\end{aligned}$$

So by Definition B(iv)-1,

$$\lim_{n\to\infty} \frac{2n + (-1)^n}{n+1} = 2,$$

as required. $\square$

Exercise B(iv)-2. Show that

$$\lim_{n\to\infty} \frac{n^2 + (-1)^n\, n}{3n^2 + 1} = \frac{1}{3}.$$

The same rules apply as in Exercise B(iv)-1. HINT: Show that

$$\left| \frac{n^2 + (-1)^n\, n}{3n^2 + 1} - \frac{1}{3} \right| \leq \frac{3n+1}{3(3n^2+1)}.$$

Since $1 \leq n$ by assumption, the latter is $\leq 4n/(3(3n^2+1)) < 4n/(3(3n^2)) = 4/(9n)$. Now proceed similarly to the proof of Proposition B(iv)-1_{EE}.

Exercise B(iv)-3. Show that

$$\lim_{n\to\infty} \frac{4n^3 + n + \sin n}{7n^3 + 3} = \frac{4}{7}.$$

The same rules apply as in Exercises B(iv)-1 and B(iv)-2.

We can prove most of the usual, familiar **limit laws** using Definition B(iv)-1 above. For example:

Proposition B(iv)-$1_{\mathbf{EEE}}$.

$$\lim_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n,$$

providing both limits on the right hand side exist.

Proof. Let $\varepsilon > 0$. Let's write

$$\lim_{n\to\infty} x_n = L \quad \text{and} \quad \lim_{n\to\infty} y_n = M. \tag{$**$}$$

[Scratch work: we want to find R so that $n > R \Rightarrow |x_n + y_n - (L+M)| < \varepsilon$. But note, by the triangle inequality ($\triangle_=$) on $\mathbb{R}$,

$$|x_n + y_n - (L+M)| = |(x_n - L) + (y_n - M)| < |x_n - L| + |y_n - M|. \tag{$\dagger$}$$

If we pick n large enough to make each of the terms $|x_n - L|$ and $|y_n - M|$ smaller than $\varepsilon/2$—and we *can* do this, by Definition B(iv)-1 and ($**$)—then we have the desired inequality. OK, here's what we write.] By ($**$) and Definition B(iv)-1, we can choose $R_1, R_2 \in \mathbb{R}$ such that $n > R_1 \Rightarrow |x_n - L| < \varepsilon/2$ and $n > R_2 \Rightarrow |y_n - M| < \varepsilon/2$. Let R be the larger of R_1 and R_2. If $n > R$, then by (†),

$$|x_n + y_n - (L + M)| \leq |x_n - L| + |y_n - M| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

So by Definition B(iv)-1,

$$\lim_{n\to\infty}(x_n + y_n) = L + M = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n,$$

as required. □

Exercise B(iv)-4. Prove that, if

$$\lim_{n\to\infty} x_n = L$$

(where L is a finite number) and $C \in \mathbb{R}$, then

$$\lim_{n\to\infty} Cx_n = C\,L.$$

The following limit law is a bit harder to prove. First we need a brief lemma.

Lemma B(iv)-1. For all real numbers x and y, we have

$$|x| - |y| \leq |x + y|.$$

Proof. Let $x, y \in \mathbb{R}$. Then by the triangle inequality ($\triangle_=$) and some algebra, we have

$$|x| = |(x + y) - y| \leq |x + y| + |-y| = |x + y| + |y|.$$

Subtracting $|y|$ from both sides gives $|x| - |y| \leq |x + y|$, as claimed. □

Now here is the limit law to which Lemma B(iv)-1 will be applied.

Proposition B(iv)-1$_{\mathbf{EEEE}}$.

$$\lim_{n\to\infty} x_n y_n = \left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} y_n\right),$$

providing both limits on the right exist.

Proof. Let's call the limits on the right L and M, as in (∗∗). Let $\varepsilon > 0$. [Scratch work: we want $\Rightarrow |x_n y_n - LM| < \varepsilon$. The trick is as follows: write $|x_n y_n - LM| = |x_n y_n - x_n M + x_n M - LM|$ and note, by the triangle inequality ($\triangle_=$) on $\mathbb{R}$, that

$$|x_n y_n - x_n M + x_n M - LM| \leq |x_n y_n - x_n M| + |x_n M - LM| = |x_n||y_n - M| + |M||x_n - L|.$$

By Definition B(iv)-1, we can pick n large enough to make $|x_n - L| < \varepsilon/(2(|M| + 1))$ and $|y_n - M| < \varepsilon/(2(|L| + 1))$. (It's important to have an $|M| + 1$ and an $|L| + 1$ in the denominators, rather than just an $|M|$ and $|L|$, since M or L could conceivably be zero, and we want to avoid dividing by zero.) We can also, for n large enough, assure that $|x_n| < |L|+1$ for the following reason: we are assuming that $\lim_{n\to\infty} x_n = L$, so by Definition B(iv)-1 with $\varepsilon = 1$ we have $|x_n - L| < 1$ for n large enough; but $|x_n| - |L| \le |x_n - L|$ by Lemma B(iv)-1, so for n large enough, we have $|x_n| - |L| < 1$, or $|x_n| < |L| + 1$.

Now it may not be clear why we are bounding the various quantities involved in the indicated ways, but it will be once we write our proof. So here's what we write.] By $(**)$ and Definition B(iv)-1, we can choose an R_1 such that $n > R_1 \Rightarrow |x_n - L| < \varepsilon/(2(|M|+1))$, an R_2 such that $n > R_2 \Rightarrow |y_n - M| < \varepsilon/(2(|L|+1))$, and R_3 such that $n > R_3 \Rightarrow |x_n| < |L| + 1$. Let $N = \max\{R_1, R_2, R_3\}$: then

$$\begin{aligned} n > R &\Rightarrow n > R_1 \text{ and } n > R_2 \text{ and } n > R_3 \\ &\Rightarrow |x_n y_n - LM| = |x_n y_n - x_n M + x_n M - LM| \le |x_n y_n - x_n M| + |x_n M - LM| \\ &= |x_n||y_n - M| + |M||x_n - L| < (|L|+1)\frac{\varepsilon}{2(|L|+1)} + |M|\frac{\varepsilon}{2(|M|+1)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

So by Definition B(iv)-1,

$$\lim_{n\to\infty} x_n y_n = LM = \left(\lim_{n\to\infty} x_n\right)\left(\lim_{n\to\infty} y_n\right),$$

as required. □

Exercise B(iv)-5. Prove that, if

$$\lim_{n\to\infty} x_n = L$$

and $L > 0$, then the x_n's are "eventually" positive, meaning there is an $R \in \mathbb{R}$ such that $n > R \Rightarrow x_n > 0$. Hint: let $\varepsilon = L/2$: by Definition B(iv)-1, there is an $R \in \mathbb{R}$ such that $n > R \Rightarrow |x_n - L| < L/2$. What does this tell you about x_n itself ?

Exercise B(iv)-6. Prove that

$$\lim_{n\to\infty} \sqrt{x_n} = \sqrt{\lim_{n\to\infty} x_n},$$

providing the limit on the right exists and is > 0. Hints: first you need an R_1 such that $n > R_1 \Rightarrow x_n > 0$ (use Exercise B(iv)-5), so that you can even consider the square roots on the left. Then, given $\varepsilon > 0$, you need an R_2 such that $n > R_2 \Rightarrow |\sqrt{x_n} - \sqrt{L}| < \varepsilon$, where $L = \lim_{n\to\infty} x_n$. To achieve the latter, use the fact that

$$\sqrt{a} - \sqrt{b} = \frac{a-b}{\sqrt{a}+\sqrt{b}}$$

for $a, b > 0$. Now let $R = \max\{R_1, R_2\}$, and go for it.

Remark. Exercise B(iv)-6 exemplifies a more general limit law, namely: if $\lim_{n\to\infty} x_n = L$ and f is *continuous* at $x = L$, then

$$\lim_{n\to\infty} f(x_n) = f(L).$$

Indeed, this is essentially the (or at least a) definition of continuity.

Next, we have the following HUGELY powerful limit law.

Proposition B(iv)-1$_{\mathbf{EEEEE}}$: The Squeeze Law. If

$$x_n \leq y_n \leq z_n \tag{$\dagger$}$$

for all n sufficiently large (meaning for all n greater than or equal to some fixed number R_1), and

$$\lim_{n\to\infty} x_n = L = \lim_{n\to\infty} z_n,$$

then

$$\lim_{n\to\infty} y_n = L$$

as well.

Proof. let R_1 be large enough that (†) holds for $n > R_1$ (such an R_1 exists by assumption). Let $\varepsilon > 0$. By Definition B(iv)-1 there is an R_2 such that $n > R_2 \Rightarrow |x_n - L| < \varepsilon$, and an R_3 such that $n > R_3 \Rightarrow |z_n - L| < \varepsilon$. But note $|x_n - L| < \varepsilon \Rightarrow -\varepsilon < x_n - L \Rightarrow L - \varepsilon < x_n$; similarly $|z_n - L| < \varepsilon \Rightarrow z_n - L < \varepsilon \Rightarrow z_n < L + \varepsilon$. let $R = \max\{R_1, R_2, R_3\}$. Then by (†),

$$n > R \Rightarrow n > R_1 \text{ and } n > R_2 \text{ and } n > R_3 \Rightarrow L - \varepsilon < x_n \leq y_n \leq z_n < L + \varepsilon.$$

So $n > R \Rightarrow L - \varepsilon < y_n < L + \varepsilon$; the latter is the same as $|y_n - L| < \varepsilon$. So by Definition B(iv)-1,

$$\lim_{n\to\infty} y_n = L,$$

as required. □

Exercise B(iv)-7. Prove that

$$\lim_{n\to\infty} x_n = L \Leftrightarrow \lim_{n\to\infty} |x_n - L| = 0.$$

(Deduce this directy, but carefully, from Definition B(iv)-1.)

The result of Exercise B(iv)-7 is quite useful; for example, we use it in the following:

Proposition B(iv)-1$_{\mathbf{EEE}}$ revisited.

$$\lim_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n,$$

providing both limits on the right hand side exist.

Proof. We've already proved this, but this time, we do so using the squeeze law instead of ε's and R's. Here's how: For all $n \geq 1$ we have, by the triangle inequality ($\triangle_=$),

$$0 \leq |x_n + y_n - (L + M)| \leq |x_n - L| + |y_n - M|. \tag{$\square$}$$

By assumption and by Exercise B(iv)-7, the right side of (□) goes to zero as $n \to \infty$; certainly the left side does too. So by the squeeze law, $|x_n + y_n - (L + M)| \to 0$ as $n \to \infty$; so by Exercise B(iv)-7,

$$\lim_{n\to\infty} (x_n + y_n) = L + M,$$

as required. □

The point is that, if one assumes the squeeze law, then one often does not need arguments that directly make use of ε's and R's.

Exercise B(iv)-8. Using the squeeze law instead of ε's and R's, prove that

$$\lim_{n\to\infty} \frac{3n + \cos n}{2n} = \frac{3}{2}.$$

Hint: get a common denominator to show that

$$0 \leq \left| \frac{3n + \cos n}{2n + 7} - \frac{3}{2} \right| \leq \frac{3}{2n}.$$

Now use the squeeze law, Exercise B(iv)-7, Proposition B(iv)-1_{E}, and Exercise B(iv)-4.

Exercise B(iv)-9. Using the squeeze law instead of ε's and R's, re-prove the result of Exercise B(iv)-4.

There are MANY MANY other "ε-R" proofs that can be done *without* the ε's and the R's, if one has the squeeze law at one's disposal. Of course the proof of the squeeze law *does* require ε's and R's, so the power of the squeeze law does *not* diminish, but in fact illuminates, the value of ε-R proofs.

Part B(v): Mathematical Induction

Suppose you want to show that a certain statement is true for any positive integer n. For example you might want to prove that, given any positive integer n,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2},$$

i.e. the sum of the first n positive integers is $n(n+1)/2$. Or you might want to show that, for any positive integer n,

$$\int_0^1 (-\ln t)^n \, dt = n!.$$

These seem like pretty hard things to prove, since there are infinitely many positive integers and you only have a finite amount of time and patience.

Well, maybe you don't actually have to look at every integer n. It's sort of like dominoes: suppose you have an infinite line of dominoes, numbered consecutively as $A(1), A(2), A(3), \ldots$, all standing on end. Also suppose:

(a) The first domino $A(1)$ is knocked over, and

(b) The dominoes are so arranged that each one, upon falling, will topple the next. That is, whenever the kth domino $A(k)$ falls, so will the $(k+1)$st domino $A(k+1)$.

It's clear, at least intuitively, that from **(a)** and **(b)** you can conclude that all dominoes will eventually fall; that is, that the nth domino $A(n)$ will topple for any integer n. The principle of mathematical induction works in just the same way, except that instead of dominoes one has *mathematical assertions.* For example, $A(n)$ could be the assertion "$1+2+3+\cdots+n = n(n+1)/2$," or "$\int_0^1 (-\ln t)^n\, dt = n!$." We have:

Principle B(v)-1: the principle of mathematical induction. Let $A(n)$ be an assertion regarding a positive integer n. To prove that $A(n)$ is true for all n, it is enough to show:

- **(Step 1)** $A(1)$ is true,
- **(Step 2)** For any positive integer k, $A(k) \Rightarrow A(k+1)$.

You don't need dominoes to understand the principle mathematical induction: think of it this way. Suppose you want to prove a statement $A(n)$ regarding an arbitrary positive integer n. If you can ascertain $A(1)$, and that $A(k)$ gives you $A(k+1)$ for each positive integer k, then you can conceptually leapfrog from $A(1)$ to $A(2)$, and then from $A(2)$ to $A(3)$, and then from $A(3)$ to $A(4)$, and so on until you conclude $A(n)$. Step 1 of mathematical induction gives you your starting point; Step 2 allows you to make all of the jumps (in your head, you make them all at once).

So we can rewrite Principle B(v)-1, as a model **proof** by mathematical induction:

Proposition B(v)-1. For any positive integer n, $A(n)$ is true.

Proof.

Step 1. [Prove $A(1)$]. So $A(1)$ is true.

Step 2. Assume $A(k)$. [Then do what you need to do, to show that]: So $A(k+1)$ follows.

Therefore, by the principle of mathematical induction, $A(n)$ is true for all positive integers n. $\square$

Step 1 is often called the "anchor" of a proof by mathematical induction. Step 2 is called the "inductive step;" the hypothesis $A(k)$ is called the "induction hypothesis." And remember: for Step 2 you need not *prove* that $A(k)$ is true; only that *whenever* $A(k)$ *is true, so is* $A(k+1)$, or in other words, that $A(k) \Rightarrow A(k+1)$.

For example, let us use mathematical induction to prove:

Proposition B(v)-1$_E$. The sum of the first n positive integers is $n(n+1)/2$.

Proof. We let $A(n)$ be the statement

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}.$$

Step 1. Is $A(1)$ is true? Yes, since $1 = 1(1+1)/2$.

Step 2. We need to show that $A(k)$ implies $A(k+1)$ for all positive integers k. The statement $A(k)$ is

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}.$$

Assume that this is true. We need to show that $A(k+1)$ follows; in other words that

$$1 + 2 + 3 + \cdots + (k+1) = \frac{(k+1)((k+1)+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

Let's examine the left-hand side of $A(k+1)$. We have

$$\begin{aligned} 1 + 2 + 3 + \cdots + (k+1) &= (1 + 2 + 3 + \cdots + k) + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}, \end{aligned} \tag{1}$$

the second equality following from the induction hypothesis. But equation (1) is just the assertion $A(k+1)$.

We have shown that $A(1)$ is true, and that $A(k)$ implies $A(k+1)$ for all positive integers k. By the principle of mathematical induction, we have proved that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

for all positive integers n. □

Note that the *key step* in the above was equation (1), where we used $A(k)$ to deduce $A(k+1)$. In particular, in equation (1), we: first wrote down the left-hand side of $A(k+1)$, then did some algebra to express this in terms of the left-hand side of $A(k)$, then used the induction hypothesis to rewrite this in terms of the *right-hand side* of $A(k)$, then did some algebra to express this in terms of the right-hand side of $A(k+1)$. This is often the kind of strategy that will work in a proof by mathematical induction.

Exercise B(v)-1. Use mathematical induction to prove that, for any positive integer n,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Exercise B(v)-2. Use mathematical induction to prove that, for any positive integer n,

$$1 + 3 + 5 + 7 + \cdots + 2n - 1 = n^2.$$

That is: the sum of the first n consecutive *odd* positive integers equals n^2.

Exercise B(v)-3. Use mathematical induction to prove that, for any positive integer n,

$$\frac{d}{dx}x^n = nx^{n-1}$$

(pretend you didn't already know this, although it's OK to assume it's true for $n = 1$). Hint: for the inductive step, use the product rule.

Exercise B(v)-4. [For students who have had second semester Calculus.] Use mathematical induction to prove that, for any positive integer n,

$$\int_0^1 (-\ln x)^n \, dx = n!.$$

Hints: (a) note that this is an improper integral (since $\ln x \to -\infty$ as $x \to 0^+$); (b) use integration by parts.

Exercise B(v)-5. Use mathematical induction to prove that, for any positive integer n, the product of any n integers of the form $4m+1$ (where m is an integer) is itself of the form $4m+1$.

Exercise B(v)-6. Let $A(n)$ be the statement

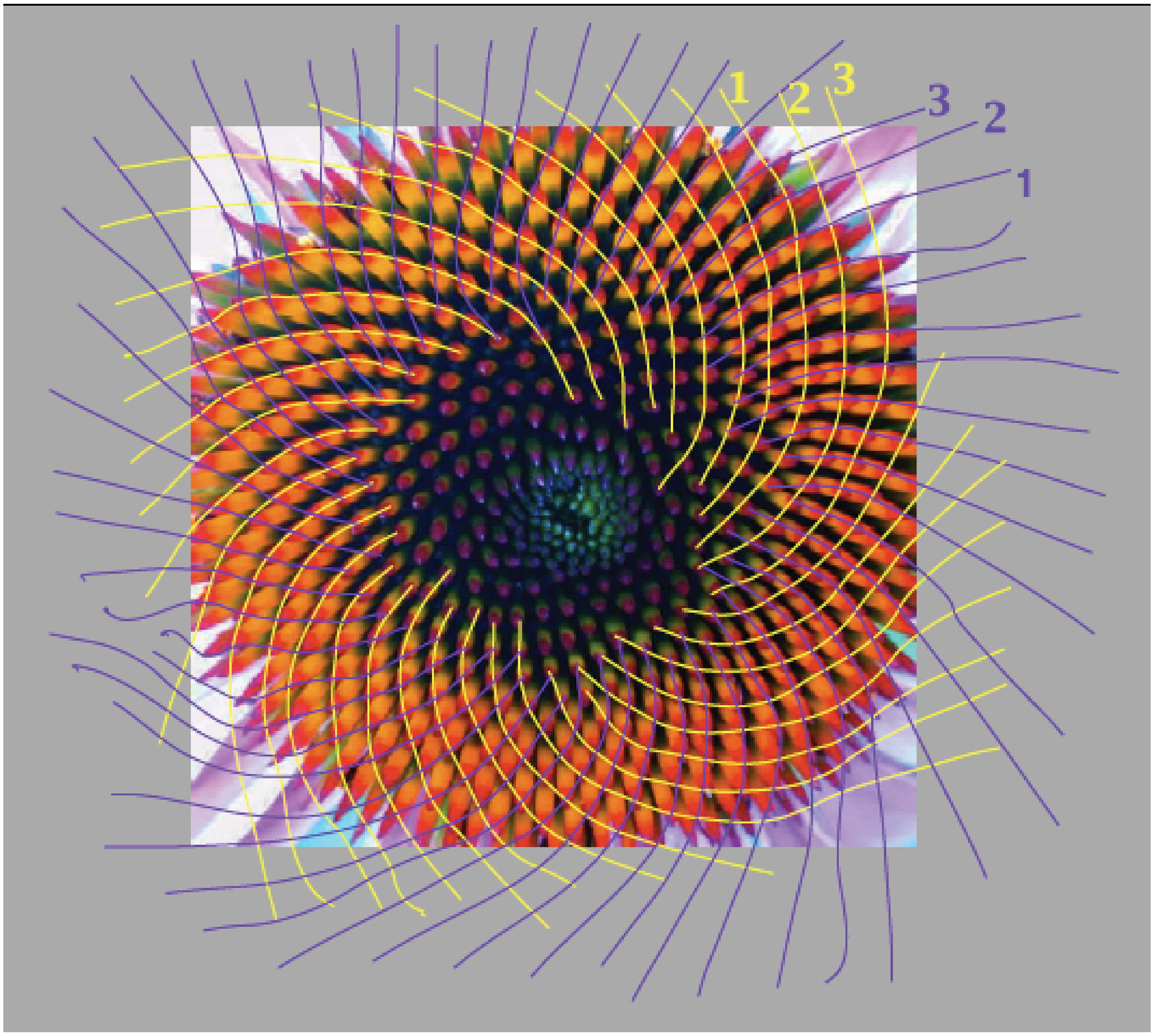
$$1 + 2 + 3 + \cdots + n = \frac{(2n+1)^2}{8}.$$

Prove that if $A(k)$ is true for any positive integer k, then so is $A(k+1)$. Is $A(n)$ true for all positive integers n? Explain your answer.

Remark. The principle of mathematical induction, while entirely plausible and perhaps even "obvious," is in fact dependent on the *well-ordering principle*, a deep fact that we have already mentioned in connection with the division algorithm. See p. 4 above, and Axiom 3.1.1, p. 105, of our course text.

Part B(vi): More on induction: Fibonacci numbers

Exercise B(vi)-1. Count the number of clockwise (yellow) and counterclockwise (purple) spirals in the coneflower below.



Clockwise spirals: ____________ Counterclockwise spirals: ____________

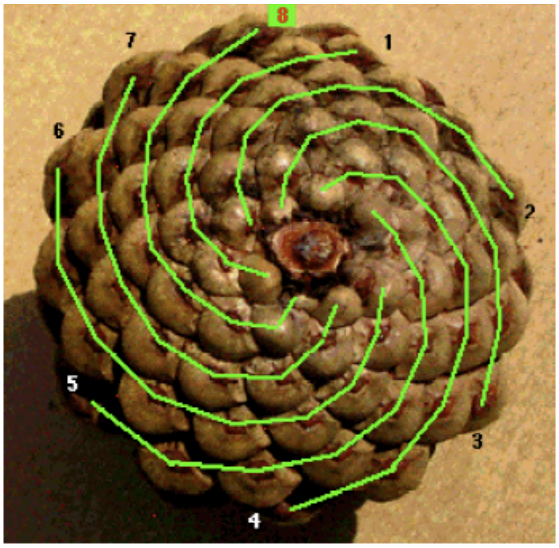
What's the significance of Exercise B(vi)-1? To answer, we define the *Fibonacci sequence* F_n, which looks like this:

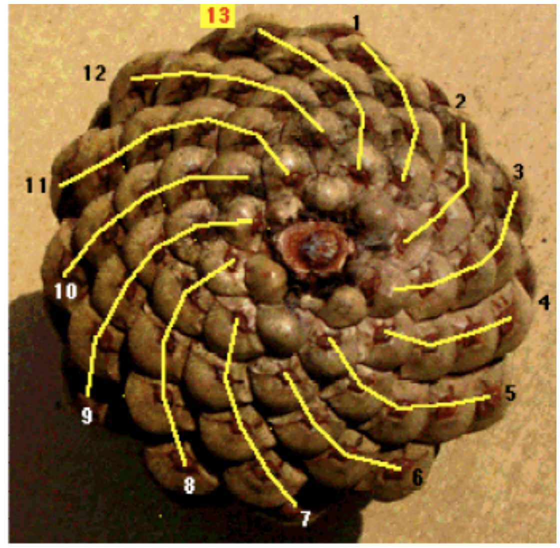
$$1, 1, 2, 3, 5, 8, 13, 21, \ldots.$$

The rule for finding terms in this sequence is: the first term is 1; the second term is 1; to get any other term, add together the previous two terms. That is: $F_1 = F_2 = 1$; $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$.

Exercise B(vi)-2. Write down the nine Fibonacci numbers (that is, the nine terms in the Fibonacci sequence) that come right after the last Fibonacci number listed above.

FACT: Fibonacci numbers are EVERYWHERE. See, for example, Exercise B(vi)-1 above. Similarly, count clockwise and counterclockwise spirals on a pine cone: you'll get consecutive Fibonacci numbers! Really!!





Similar things happen with sunflowers, pineapples, broccoli florets, etc. See

`https://en.wikipedia.org/wiki/Fibonacci_number`

Fibonacci numbers satisfy many curious relations. Here's one.

Exercise B(vi)-3. Using the principle of mathematical induction, prove that

$$F_{n+3}F_n - F_{n+1}F_{n+2} = (-1)^{n+1}.$$

Hint: If the above statement is $A(n)$, then $A(k+1)$ is the statement

$$F_{k+4}F_{k+1} - F_{k+2}F_{k+3} = (-1)^{k+2}.$$

To see how this can be obtained from $A(k)$, rewrite F_{k+4} and F_{k+2}, in the above statement of $A(k+1)$, using the fact that a given Fibonacci number equals the sum of its two predecessors.

Particularly interesting things happen when we examine *ratios* of successive Fibonacci numbers. Let's do this.

Exercise B(vi)-4. Define a sequence R_n by

$$R_n = \frac{F_{n+1}}{F_n}$$

for $n \geq 1$ (F_n denotes the nth Fibonacci number, as above). So the sequence R_n starts like this:

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \ldots.$$

Write down the next nine R_n's as fractions. Then write these nine terms as decimal numbers, with at least 4 places after the decimal point. Do the R_n's appear to be converging? That is, do they appear to have a limit? If so, what (approximately) does this limit appear to be (to as many decimal places as you care to speculate)?

The number to which your above R_n's converge is, actually, a number that shows up in various other places too.

In the next problem, we investigate one of those places.

Exercise B(vi)-5. Find a half-dozen single-switch switchplates, meaning this kind of thing:

around your home, at school, etc. (Avoid switchplates that have extra stuff like electrical outlets, or that have multiple switches, or non-rectangular shapes, etc. Also, try not to take all of your switchplates from the same home, or classroom, etc.) Measure the height and width of each switchplate in millimeters. Then compute the *Proportion* of the switchplate, defined to be the ratio of the height (longer side) to width (shorter side). Do this for each of your six switchplates, and supply supply the relevant info below (if your computed answers have more than four digits after the decimal, it suffices to write down only the first four of these digits):

Height: ____________ Width:____________ Proportion:____________

Height: ____________ Width:____________ Proportion:____________

Height: ____________ Width:____________ Proportion:____________

Height: ____________ Width:____________ Proportion:____________

Height: ____________ Width:____________ Proportion:____________

Height: ____________ Width:____________ Proportion:____________

The average (mean) of the above six Proportions is: ____________

Exercise B(vi)-6. The number $(1+\sqrt{5})/2$, often called the *golden mean* or the *golden ratio*, and often denoted by Φ, is special. It shows up in many real-life, and mathematical, situations. What are some such situations? To answer, plug this number into your calculator, and evaluate as a decimal to a few decimal places. How does what you get compare to some of the numbers above? See especially exercises B(vii)-3 and B(vii)-4.

Remark. The golden ratio Φ, or numbers close to it, also show up when you divide your height by the height of your belly button; the height of your face (chin to crown) by the width of your face; etc. (Try it!!) There's an awful lot of debate as to whether these phenomena are deeply significant or not. (Perhaps the debate itself makes them significant.)

Let's return to the study of Fibonacci numbers *per se*. We note that the formula $F_{n+2} = F_{n+1} + F_n$ for these numbers is *recursive*; it expresses a given Fibonacci number in terms of previous ones. Recursive formulas, on their own, can be a bit of a pain, because you can only use them to figure out a given term if you have already worked out all terms coming *before* that given one. (To compute F_{n+2} you only need, on the surface, to know F_n and F_{n+1}, but of course, to know these latter two quantities you need to have computed F_{n-1} and F_{n-2}, and so on down the line.)

There are various methods that can sometimes be employed to turn recursive formulas into *closed* formulas. A closed formula for a sequence is one where each term a_n is expressed directly in terms of the integer n, and not in terms of a_{n-1}, a_{n-2}, etc.

As it turns out, there *is* a convenient closed formula for Fibonacci numbers. We won't discuss the derivation of this formula, but we will state the formula, and *prove* that the formula is correct.

Proposition B(vi)-1. If F_n denotes the nth Fibonacci number, then

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

Proving this proposition is quite straightforward, if one has the following generalization of the principle of mathematical induction.

Principle B(vi)-1: the principle of "double whammy" mathematical induction, or DWMI. Let $A(n)$ be an assertion regarding a positive integer n. To prove that $A(n)$ is true for all n, it is enough to show:

- **(Step 1)** $A(1)$ is true,
- **(Step 2)** $A(2)$ is true,
- **(Step 3)** Whenever $A(k)$ and $A(k+1)$ are true for a positive integer k, then so is $A(k+2)$.

Exercise B(vi)-7. Come up with an interesting, convincing ANALOGY for DWMI. Something like our domino analogy for the original principle of mathematical induction, but that better suits the situation at hand. be creative!!

Exercise B(vi)-8. Use DWMI to prove Proposition B(vi)-1. Hint: it might be useful to note that

$$\left(\frac{1 \pm \sqrt{5}}{2}\right)^2 = \frac{1 \pm 2\sqrt{5} + \sqrt{5}^2}{4} = \frac{6 \pm 2\sqrt{5}}{4} = \frac{3 \pm \sqrt{5}}{2}.$$

Exercise B(vi)-9. Let R_n be the ratio defined in Exercise B(vi)-4. Prove that

$$\lim_{n\to\infty} R_n = \Phi$$

where, again, Φ denotes the golden ratio, $\Phi = \frac{1+\sqrt{5}}{2}$. (You don't need to use fancy "ε-R" limit arguments, like those in Section B(iv) above, to do this. Just use standard calculus techniques.) Does this confirm the observations you made based on numerical calculations you did in Exercise B(vi)-6?

Part B(vii): Proof by contradiction

We've demonstrated various different ways of proving various types of statements. Note that any given statement may be amenable to more than one strategy of proof.

We now introduce one more proof strategy, namely, the strategy of *proof by contradiction.*

Here's the big idea behind this strategy: suppose you want to prove a statement T. If the assumption of $\sim T$ leads to an *absurdity*, meaning something that is logically impossible, then the assumption of $\sim T$ must have been incorrect, whereby $\sim(\sim T)$, which is to say T, must follow.

The absurdity that one often shoots for in a proof by contradiction is one of the form "V and $\sim V$," where V is *any statement whatsoever!!* Indeed a statement, regardless of its nature, cannot be true at the same time as its negation is, so "V and $\sim V$" is always an absurdity.

In sum, the general idea is:

Proposition B(vii)-1. T.

Proof. Assume $\sim T$. [Then do some stuff to conclude:] Therefore, V.

[Then do some other stuff to conclude:] Therefore, $\sim V$.

Thus, V and $\sim V$. Contradiction. Therefore, T. □

Sometimes, either V or $\sim V$ will be obvious. For example:

Proposition B(vii)-1$_{\mathbf{E}}$. There are no integers a and b with $6a + 28b = 1$.

Proof. Let T be the statement of this proposition. We assume $\sim T$ to be true; that is, we assume that there *do* exist integers a and b with $6a + 28b = 1$. Now 2 divides 6 and 2 divides 28, so by Exercise B(i)-3 above, 2 divides $6a + 28b$ for any integers a and b, so by the assumption $\sim T$, 2 divides 1, meaning 1 is even.

But 1 is odd. Contradiction. So our assumption $\sim T$ must be false, so T is true. That is: there are no integers a and b with $6a + 28b = 1$. □

Exercise B(vii)-1. Use proof by contradiction to show that there are no integers a and b with $6a + 21b = 1$.

Exercise B(vii)-2. Use proof by contradiction to show that there are no integers a and b such that a and b are both odd, and $a^2 + b^2$ is a perfect square. Hint: assume the statement to be proved is false, so that $a^2 + b^2 = c^2$ for some integers a, b, and c, with a and b both odd. Conclude that $c^2 - 2$ is divisible by 4. Then derive a contradiction using Exercise B(i)-2 above (and the division algorithm).

As a less simple example, we have:

Proposition B(vii)-1$_{\mathbf{EE}}$. There are infinitely many prime numbers.

Proof. Assume it is not the case that there are infinitely many prime numbers: that is, assume there are finitely many, say K, of them. Denote them by $p_1, p_2, \ldots, p_K$.

Put $M = p_1 p_2 \cdots p_K + 1$, and let p be a prime number that divides M. (Every integer greater than 1 has a prime divisor.) Since p equals one of the primes $p_1, p_2, \ldots, p_K$ (as these are all of the primes), p certainly divides the product of all these primes: so p also divides $N = p_1 p_2 \cdots p_K$. But any integer dividing two integers divides their difference, so p divides $M - N$.

On the other hand, by definition of $M - N$, we have

$$M - N = (p_1 p_2 \cdots p_K + 1) - p_1 p_2 \cdots p_K = 1.$$

But 1 is not divisible by any prime, so p cannot divide $M - N$.

So $p|(M - N)$ and $p \nmid (M - N)$. Contradiction. So there are infinitely many prime numbers. □

Exercise B(vii)-3. Prove that there are infinitely many positive prime numbers of the form $4\ell + 3$ (for ℓ an integer). Hint: Assume this is not the case. That is, assume there are finitely many, say K, positive prime numbers of the form $4\ell + 3$. Denote them by $p_1, p_2, \ldots, p_K$, and let $S = \{1, p_1, p_2, \ldots, p_K\}$. Now put $M = 4p_1 p_2 \cdots p_K - 1$. Note that M is of the form $4j + 3$, for j an integer (why?) Now proceed in a manner similar to that of Proposition B(vii)-1$_{\text{EE}}$ above. At some point, you may want to use the result of Exercise B(v)-5.

Remark. Suppose V is some statement such that $V \Rightarrow \sim V$ and $\sim V \Rightarrow V$. Well, note that either V or $\sim V$ must be true. In the first case we can deduce $\sim V$, and in the second we can deduce V. In either case, we find V *and* $\sim V$ are true.

IN OTHER WORDS: one way of arriving at the statement "$V \Rightarrow \sim V$ and $\sim V \Rightarrow V$" used in the proof of Proposition B(vii)-1 is to come up with some statement V such that $V \Rightarrow \sim V$ and $\sim V \Rightarrow V$. The corresponding proof by contradiction will then look like this:

Proposition B(vii)-2. T.

Proof. Assume $\sim T$. Assume V. [Then do some stuff to conclude:] Therefore, $\sim V$.

Assume $\sim V$. [Then do some stuff to conclude:] Therefore, V.

In either case (V or $\sim V$), we have V and $\sim V$. Contradiction. Therefore, T. ☐

For example, we have:

Proposition B(vii)-2$_\mathbf{E}$. The square of any real number is non-negative (that is, ≥ 0).

Proof. Let T be the statement of the proposition. Assume $\sim T$. That is, assume there is some real number b with

$$b^2 < 0. \tag{1}$$

To derive a contradiction to $\sim T$, we're going to consider the statement V: $b > 0$. Note that $\sim V$ is the statement $b \leq 0$. We'll show that, if we assume $\sim T$, then $V \Rightarrow \sim V$ and $\sim V \Rightarrow V$. This will tell us, as described above, that the assumption $\sim T$ must have been false.

So assume V: $b > 0$. Then, since multiplying both sides of an inequality by a positive number preserves the direction of the inequality, we can multiply both sides of equation (1) by b^{-1} to get $b < 0$, which certainly implies $b \leq 0$. So $\sim V$ is true.

Now assume $\sim V: b \leq 0$. Of course b can't be zero because of equation (1) (and the fact that $0^2 = 0$), so $b < 0$. Then, since multiplying both sides of an inequality by a negative number reverses the direction of the inequality, we can multiply both sides of equation (1) by b^{-1} to get $b > 0$. So V is true.

In either case ($b > 0$ or $b \leq 0$), we have $b > 0$ and $b \leq 0$. Contradiction. So the square of any real number is non-negative. ☐

We're not claiming that the contradiction method gives the *easiest* proof of Proposition B(vii)-2$_\mathrm{E}$. But it does give *a* proof, and one that illustrates the ideas behind Proposition B(vii)-2.

Our next exercise provides a perhaps meatier illustration and application of these ideas, and is enough to make your head spin. (If your head is spinning already, this exercise is enough to start it spinning in the opposite direction, simultaneously.)

To present this exercise, we recall a couple of mathematical ideas: first, for any sets X and Y (finite or not), we say X and Y are *equivalent* if there is a bijection (a one-to-one, onto function) from X to Y. (Equivalent sets are said to have the same *cardinality*. Very roughly, cardinality can be understood as a measure of the *size* of a set.)

Next: for any set X, the *power set* $\mathcal{S}(X)$ is defined to be the set of all *subsets* of X.

We have:

Exercise B(vii)-4. Fill in the blanks below to prove the following proposition: no set is equivalent to its power set.

Proof. We want to prove T: no set is equivalent to its power set. So assume $\sim$_________, that is, suppose *some* set, call it X, *is* equivalent to its power set $\mathcal{S}(X)$. This means there is a one-to-one, onto function f from X to _________.

By definition of f we know that, for each element x of X, $f(x)$ belongs to $\mathcal{S}(X)$, so $f(x)$ is a _________ of X. This subset might contain the element x, or it might not. Let's consider the cases where it doesn't. Specifically, let's consider the set B of all elements of X such that $f(x)$ *does not* contain x. That is, let

$$B = \{x \in X : \underline{\hspace{2cm}} \notin f(x)\}. \tag{2}$$

Since B is a subset of _________, we have $B \in \mathcal{S}(X)$, and therefore, since f takes X *onto* $\mathcal{S}(X)$, there is some $y \in X$ with

$$f(y) = B. \tag{3}$$

Consider the statement $V : y \in B$. We are going to show that, under the assumption $\sim T$, we have $V \Rightarrow \sim V$ and $\sim V \Rightarrow V$. Then, as in Proposition B(vii)-2, we will have a __________________ to the statement $\sim T$, and thus we will be able to conclude the statement _________, thereby proving our proposition.

So assume $y \in B$. By the definition (2) of B, this means $y \notin f(y)$. But again, by equation (3), we have $f(y) =$_________. So $y \notin B$.

Now assume $y \notin B$. By the definition (2) of B, this means $y \in$ _________. But again, by equation (3), we have $f(y) = B$. So $y \in B$.

In either case ($y \in B$ or __________________), we have __________________ and $y \notin B$. Contradiction. Therefore, no set is equivalent to its __________________. $\square$