Probability: what it is.

Definition: probability is the study of probability models.

A probability model is:

A) A sample space S, together with

B) A way of assigning a number $P(E)$, called the probability of E, to each event (that is, subset) of S.

C) Some axioms (assumptions):

Axiom 1: $0 \leq P(E) \leq 1$ for any event E.

Axiom 2: $P(S) = 1$.

Axiom 3: Suppose $E_1, E_2, E_3, \ldots$ is a (perhaps infinite) list of events, that are mutually exclusive (that is, no two can happen together, meaning $E_i E_j = \emptyset$ for all $i \neq j$). Then

$$P(E_1 \cup E_2 \cup E_3 \cup \ldots) = P(E_1) + P(E_2) + P(E_3) + \ldots$$

Example 1:

For a certain unfair, 6-sided die, $P(\text{even}) = 2P(\text{odd})$. If all odd #'s are equally likely, as are all powers of 2 (including

$1 = 2^0$),

find $P(\{i\})$ for $1 \le i \le 6$.

Solution.

First, we find P(even) and P(odd). Since even $\cup$ odd = {all outcomes}, we have

$$P(\text{even} \cup \text{odd}) = 1$$

by Axiom 2, so by Axiom 3,

$$P(\text{even}) + P(\text{odd}) = 1.$$

So by assumption

$$2P(\text{odd}) + P(\text{odd}) = 1$$
$$3P(\text{odd}) = 1$$
$$P(\text{odd}) = 1/3.$$

Now by Axiom 3,

$$P(\text{odd}) = P(\{1\}) + P(\{3\}) + P(\{5\}).$$

The left side = 1/3, and all terms on the right are the same, so each one equals $1/3 \cdot 1/3 = 1/9$.

But all powers of 2 are equally likely, so $P(\{2\}) = P(\{4\}) = P(\{1\}) = 1/9$.

Finally, by Axioms 2 and 3,

$$P(\{6\}) + P(\text{not } 6) = 1.$$

But by Axiom 3,

$$P(\text{not } 6) = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = 5/9.$$

So $P(\{6\}) = 1 - 5/9 = 4/9$.

Note: we used the fact that, for any event E, $E \cup E^c = S$, so by Axiom 2,

$P(E \cup E^c) = 1$, so by Axiom 3,

$$P(E) + P(E^c) = 1.$$

In other words, we have:

Proposition 4.1 For any event E,

$$P(E^c) = 1 - P(E).$$

Some other consequences of the axioms (proofs omitted):

Proposition 4.2
If $E \subset F$, then $P(E) \leq P(F)$.

Proposition 4.3 For any (not necessarily mutually exclusive) events E and F,

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

More generally:

Proposition 4.4. For any events E, F, G,

$$P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG) + P(EFG).$$

(Actually, this generalizes to n events, but never mind.)