EXAM 2: COMPLETELY RANDOM PRACTICE PROBLEMS

1. (a) Express the negation of the statement $\forall x \in X, \exists y \in Y : Q(x,y)$ in terms of the negation $\sim Q(x,y)$ of $Q(x,y)$. (Here, $Q(x,y)$ is some statement involving objects x,y.)

$\exists x \in X, \forall y \in Y :\sim Q(x,y)$.

(b) Express the negation of the statement $\forall x \in X, \exists y \in Y, \forall z \in Z : Q(x,y,z)$ in terms of $\sim Q(x,y,z)$. (Here, $Q(x,y,z)$ is some statement involving objects x,y,z.)

$\exists x \in X, \forall y \in Y, \exists z \in Z :\sim Q(x,y,z)$.

(c) One way of defining limits, which you'll come across if you take MATH 3001: Introduction to Analysis, is as follows: we say

$$\lim_{n\to\infty} x_n = L$$

if

$$\forall \varepsilon > 0, \exists R \in \mathbb{R}, \forall n > R, |x_n - L| < \varepsilon. \qquad (*)$$

Given this definition, how would you express the fact that

$$\lim_{n\to\infty} x_n \neq L$$

in terms of quantifiers and a certain quantity being $\geq \varepsilon$?

$\exists \varepsilon > 0, \forall R \in \mathbb{R}, \exists n > R, |x_n - L| \geq \varepsilon$.

2. More quantifiers. Identify each of the following statements as true or false (circle "**T**" or "**F**"). **Please explain your answers:** If a statement is true, explain why (you don't need to provide a complete proof; just a sentence or two will do). If a statement is false, provide a counterexample to the statement, and explain why it's a couterexample.

(a) $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}\colon (m-n)|k$. **T** **F**

Given m and n, let $k = m - n$. Then certainly $(m-n)|k$.

(b) $\exists k \in \mathbb{Z}\colon \forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, (m-n)|k$. **T** **F**

Let $k = 0$. Then for any $m,n \in \mathbb{Z}$, $(m-n)|k$, since everything divides 0.

(c) $\sim\big(\forall m \in \mathbb{Z}, \exists k \in \mathbb{Z}\colon \forall n \in \mathbb{Z}, (m-n)|k\big)$. **T** **F**

Consider the statement $\forall m \in \mathbb{Z}, \exists k \in \mathbb{Z}\colon \forall n \in \mathbb{Z}, (m-n)|k$. This statement is true because, given m, let $k = 0$. Then for any n, $(m-n)|k$ since, again, everything divides 0. So the negation of this statement is false.

3. Let $x,y,z \in \mathbb{Z}$. Use the principle of mathematical induction to prove that, for all $n \in \mathbb{N}$,

$$z|(x-y) \Rightarrow z|(x^n - y^n).$$

Hint: you may use the fact that, for $x,y \in \mathbb{Z}$ and $k \in \mathbb{N}$,

$$x^{k+1} - y^{k+1} = x(x^k - y^k) + y^k(x-y).$$

Please clearly identify your base step, induction hypothesis, inductive step, and the conclusion of your proof.

Proof.

Let $x,y,z \in \mathbb{Z}$, and let $A(n)$ be the statement in question.

Step 1: Is $A(1)$ true? Yes, because if $z|(x-y)$, then certainly $z|(x^1 - y^1)$.

Step 2: Assume $A(k)$:

$$z|(x-y) \Rightarrow z|(x^k - y^k).$$

Now suppose $z|(x-y)$. Note that

$$x^{k+1} - y^{k+1} = x(x^k - y^k) + y^k(x-y).$$

But $z|(x^k - y^k)$ by the induction hypothesis, and $z|(x-y)$ by assumption. So by Exercise B(i)-3 in S-POP, $z|\big(x(x^k - y^k) + y^k(x-y)\big)$, so $z|(x^{k+1} - y^{k+1})$. So $A(k+1)$ follows.

So, by the principle of mathematical induction, $A(n)$ is true for all $n \in \mathbb{N}$. □

4. Use the principle of mathematical induction to prove the following:

Proposition.

For any natural number $n \geq 3$,

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n-1}{2} = \binom{n}{3}.$$

Please clearly identify your base step, induction hypothesis, inductive step, and the conclusion of your proof. Hint: you should start at $n = 3$ instead of $n = 1$. (So your base step should entail showing that the appropriate statement $A(3)$ is true.) Also, you may want to use the following formula, which you proved in a homework assignment:

$$\binom{k}{j} + \binom{k}{j-1} = \binom{k+1}{j}, \qquad (*)$$

for $j,k \in \mathbb{Z}$ and $1 \leq j \leq k$. (You DON'T need to prove $(*)$.)

Proof.

Let $A(n)$ be the statement in question.

Step 1: Is $A(3)$ true?

$$\binom{2}{2} = \binom{3}{3}$$

(since $1 = 1$), so $A(3)$ is true.

Step 2: Assume

$$A(k): \quad \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{k-1}{2} = \binom{k}{3}.$$

Then

$$\begin{aligned}&\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\cdots+\binom{k}{2}\\&=\left(\binom{2}{2}+\binom{3}{2}+\binom{4}{2}+\cdots+\binom{k-1}{2}\right)+\binom{k}{2}\\&=\binom{k}{3}+\binom{k}{2}=\binom{k+1}{3},\end{aligned}$$

the last step by $(*)$. So $A(k) \Rightarrow A(k+1)$.

So, by the principle of mathematical induction, $A(n)$ is true for all $n \geq 3$. ☐

5. Use the principle of mathematical induction to prove that, for any $n \in \mathbb{N}$,

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1.$$

(Hint: $(k+2)(k+1)! = (k+2)!$.) Please clearly identify your base step, induction hypothesis, inductive step, and the conclusion of your proof.

Proof. Let $A(n)$ be the statement in question.

Step 1: Is $A(1)$ true?

$$\begin{aligned}1 \cdot 1! &\stackrel{?}{=} (1+1)! - 1\\ 1 &= 1,\end{aligned}$$

so $A(1)$ is true.

Step 2: Assume

$$A(k): \ 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + k \cdot k! = (k+1)! - 1.$$

Then

$$\begin{aligned}&1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + (k+1) \cdot (k+1)!\\&= \big(1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + k \cdot k!\big) + (k+1) \cdot (k+1)!\\&= (k+1)! - 1 + (k+1) \cdot (k+1)!\\&= (1+k+1)(k+1)! - 1\\&= (k+2)(k+1)! - 1 = (k+2)! - 1,\end{aligned}$$

so $A(k+1)$ follows.

So, by the principle of mathematical induction, $A(n)$ is true for all n.

6. Imagine flipping a coin seven times.

(a) How many possible outcomes are there? (Think of an outcome as being a list, of length 7, of H's and T's – e.g. HTTHHTT – where the first letter in the list designates how the coin landed the first time, and so on).
$2^7 = 128$.

(b) How many possible outcomes are there in which the first *or* the fifth coin lands heads (or both)? There are no restrictions here on how the other five coins might land.

$$2^6 + 2^6 - 2^5 = 96.$$

(c) How many outcomes have *exactly* two heads, which are consecutive (right next to each other)?
6.

(d) How many outcomes have two consecutive (right next to each other) heads, with no restrictions here on how the other five coins might land?
$6 \cdot 2^5 = 192$.

(e) How many outcomes have exactly two coins landing heads?

$$\binom{7}{2} = 21.$$

7. In this problem, we'll count, in two different ways, the length-8 lists can be made from the digits 1,2,3,4,5,6,7,8, without repetition, if no odd number is next to another odd number and no even number is next to another even number. (That is: the even and odd numbers alternate.) Of course, we should get the same answer both ways.

Here are the two methods.

(a) Count as follows: how many choices are there for the first digit? Then, given that evens and odds must alternate, how many choices are there for the next digit? And then the next (again, given that evens/odds alternate)? And the next... down to the last.

$$8 \cdot 4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 1{,}152.$$

(b) This time, first count the ways of arranging the four odd numbers separately, and then count the ways of arranging four even numbers separately, then count the ways of zipping the list of odds together with the list of evens so that evens and odds alternate. (Careful: there's not just one way to zip a given odd list with a given even list.)
There are 4! ways of constructing the list of odds, and 4! ways of constructing the list of evens. Once these two lists are constructed, there are 2 ways of zipping them together (either an odd comes first, or an even comes first). So there are

$$4! \cdot 4! \cdot 2 = 1{,}152$$

possible alternating lists of evens and odds.

8. This problem concerns lists made from the letters A, B, C, D, E, F, G, H, I.

(a) How many length-6 lists can be made from these letters if repetition is not allowed and the list must contain (exactly) one D?

The number of ways of doing this if we assume that D is the first letter is: $1 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$ (one choice for the first letter – since it must be D – leaving 8 choices for the next, 7 for the next, etc.). If we assume that D is the second letter, the number of ways is: $8 \cdot 1 \cdot 7 \cdot 6 \cdot 5 \cdot 4$ (eight choices for the first letter, 1 for the second, 7 for the next, etc.). We get the same product, but in different order, for each of the 6 places that can hold a D. So the total is

$$6 \cdot (1 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4).$$

(b) How many length-5 lists can be made from these letters if repetition *is* allowed – that is, the same letter can appear more than once – but no two *consecutive* letters can be the same?

There are nine choices for the first letter. The second letter must be different from the first, so there are eight choices for the second. The third letter must be different from the second (but could be the same as the first), so there are eight choices for the third. And so on, yielding a total of

$$9 \cdot 8 \cdot 8 \cdot 8 \cdot 8 = 9 \cdot 8^4$$

possible lists.

9. Five cards are dealt off of a standard 52-card deck and lined up in a row. How many such line-ups (lists) of cards are there in which all cards have *different* face values? (Recall that a standard deck has thirteen face values – Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K – and four cards of each face value.)

$$52 \cdot 48 \cdot 44 \cdot 40 \cdot 36$$

(There are 52 choices for the first card, leaving 48 for the next, leaving 44 for the next, and so on).