1. Quantifiers.

(a) The quantifier "$\forall$" means "for all," or "for each," or "for every."
If X is a set and $Q(x)$ is a statement about a quantity x, then the statement

$$\forall x \in X : Q(x)$$

means the statement $Q(x)$ is true for every x in X.

(b) The quantifier "$\exists$" means "for some," or "for at least one," or "there exists."
If X is a set and $Q(x)$ is a statement about a quantity x, then the statement

$$\exists x \in X : Q(x)$$

means the statement $Q(x)$ is true some (at least one, possible more) x in X.

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2. Counting.

(a) Multiplication principle: if there are m ways of doing Thing 1 and, for each of these ways, there are n ways of doing Thing 2, then there are mn ways of doing Thing 1 and Thing 2 together.

Corollary: the number of length-k lists that can be made from n items is

- n^k if repetition is allowed;
- $P(n,k) = n(n-1)(n-2)\cdots(n-k+1) = \dfrac{n!}{(n-k)!}$ if not.

(b) Subtraction principle: the number of lists, or sets, with a property P equals the total number of possible lists, or sets, minus the number of lists, or sets, without property P.

(c) Addition principle: if there are m ways of doing Thing 1 and n ways of doing Thing 2, then there are $m+n$ ways of doing Thing 1 *or* Thing 2 (or both), provided you're not counting twice.

(d) Inclusion-exclusion principle: in general (that is, even if you are counting twice), if there are m ways of doing Thing 1 and n ways of doing Thing 2, then the number of ways of doing Thing 1 *or* Thing 2 (or both) is $m+n$ minus the number of ways of doing Thing 1 *and* Thing 2 together.

(e) The number of k-elements subsets of a set with n elements is

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

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3. Proof by the principle of mathematical induction.

Theorem. $\forall n \in \mathbb{N},\ A(n)$.

Proof. Step 1: Is $A(1)$ true? [Now do what you need to conclude:] So $A(1)$ is true.

Step 2: Assume $A(k)$. [Now do what you need to conclude:] So $A(k+1)$ follows. So $A(k) \Rightarrow A(k+1)$.

Therefore, by the principle of mathematical induction, $A(n)$ is true $\forall n \in \mathbb{N}$. $\square$

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4. Basic set definitions. Given sets A and B, and a universe U that contains all sets in question, we define:

(a) $A \cup B = \{x \in U : x \in A \text{ or } x \in B\}$.

(b) $A \cap B = \{x \in U : x \in A \text{ and } x \in B\}$.

(c) $A - B = \{x \in A : x \notin B\}$.

(d) $A \times B = \{\text{ordered pairs } (x, y) : x \in A \text{ and } y \in B\}$.

(e) $\overline{A} = U - A$.

(f) $\mathscr{P}(A) = \{\text{all subsets of } A\}$.

(g) $\left|P(A)\right| = 2^{|A|}$ for any set A.

(h) The statement $A \subseteq B$ is equivalent to the statement $x \in A \Rightarrow x \in B$.

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5. Intersection and union of indexed sets. Given an indexing set I and a set A_α for each $\alpha \in I$, and a universe U, we define

(a) $\bigcup_{\alpha \in I} A_\alpha = \{x \in U : x \in A_\alpha \text{ for some } \alpha \in I\}$.

(b) $\bigcap_{\alpha \in I} A_\alpha = \{x \in U : x \in A_\alpha \text{ for all } \alpha \in I\}$.

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6. Proof templates.

(a) $P \Rightarrow Q$, direct proof.

Theorem. $P \Rightarrow Q$.
Proof. Assume P. [Now do what you need to conclude:] Therefore, Q.
So $P \Rightarrow Q$. $\square$

(b) $P \Rightarrow Q$, contrapositive proof.

Theorem. $P \Rightarrow Q$.
Proof. Assume $\sim Q$. [Now do what you need to conclude:] Therefore, $\sim P$.
So $P \Rightarrow Q$. $\square$

(c) $P \Leftrightarrow Q$.

Theorem. $P \Leftrightarrow Q$.
Proof. Assume P. [Now do what you need to conclude:] Therefore, Q.
So $P \Rightarrow Q$.
Next, assume Q. [Now do what you need to conclude:] Therefore, P.
So $Q \Rightarrow P$.
Therefore, $P \Leftrightarrow Q$. $\square$

(d) $A \subseteq B$.

Theorem. $A \subseteq B$.
Proof. Assume $x \in A$. [Now do what you need to conclude:] Therefore, $x \in B$.
So $A \subseteq B$. $\square$

(e) $A = B$.

Theorem. $A = B$.
Proof. Assume $x \in A$. [Now do what you need to conclude:] Therefore, $x \in B$.
So $A \subseteq B$.
Now assume $x \in B$. [Now do what you need to conclude:] Therefore, $x \in A$.
So $B \subseteq A$.
Therefore, $A = B$. $\square$

(f) Proof by counterexample. To prove that a statement is false, you need only find one instance where the statement fails.

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7. Some special sets.

(a) $\mathbb{Z} = \{\text{integers}\} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

(b) $\mathbb{N} = \{\text{natural numbers}\} = \{1, 2, 3, \ldots\}$.

(c) $\mathbb{R} = \{\text{real numbers}\} = (-\infty, \infty)$.

(d) $\mathbb{Q} = \{\text{rational numbers}\} = \{\text{fractions } m/n : m, n \in \mathbb{Z} \text{ and } n \neq 0\}$.

(e) Let $a, b \in \mathbb{Z}$. We write $a + b\mathbb{Z}$ for the set $\{a + bm : m \in \mathbb{Z}\}$.

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8. Facts about integers.

(a) Let $a, b \in \mathbb{Z}$. We say a divides b, written $a|b$, if $b = na$ for some $n \in \mathbb{Z}$.

(b) (Division algorithm.) Given integers a and b with $b > 0$, there exist unique integers q and r for which $a = qb + r$ and $0 \leq r < b$.