MATH 2001-004: Intro to Discrete Math

November 6, 2024

Second In-class Midterm Exam

I have neither given nor received unauthorized assistance on this exam.

Name: SOLUTIONS

Signature:______________________________

Please show all work.

Please write neatly. If it's unreadable, it's ungradeable.

Use the backs of pages if you need more space.

If you get stuck on a problem, move on, and then come back to it.

Take a deep breath. Good luck!

DO NOT WRITE IN THIS BOX!

Problem	Points	Score
1	14 pts	
2	15 pts	
3	15 pts	
4	16 pts	
5	20 pts	
6	20 pts	
TOTAL	100 pts	

1. (14 points; 7 points each)

For this problem, let S be the set consisting of all lower-case English letters together with all upper-case English letters. So S has 52 distinct elements. (But this problem is the same for any set of 52 distinct elements.)

(a) **Without using a calculator**, express the number of length-5 *lists*, without repetition, that can be made from S, as a product of 5 separate integers. (That is, your answer should look something like $4 \cdot 26 \cdot 103 \cdot 5 \cdot 6$, though this is not the correct answer.)

$$52 \cdot 51 \cdot 50 \cdot 49 \cdot 48.$$

(b) **Without using a calculator**, express the number of size-5 *subsets* of S as a product of 5 separate integers. If it helps, you can use the facts that $50/5 = 10$ and $48/(4 \cdot 3 \cdot 2) = 2$. (You can use your calculator to check your work, but please show, explicitly, all the calculations you used to get your answer without a calculator.)

$$\binom{52}{5} = \frac{52!}{5!\,47!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 52 \cdot 51 \cdot 10 \cdot 49 \cdot 2.$$

2. (15 points; 5 points each) Form the negation of each of the following statements. Your negated statement should not contain the symbol "∼" in it anywhere.

(a) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \forall z \in \mathbb{R} \colon |x - y| < z \cdot \varepsilon.$

$\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, \exists z \in \mathbb{R} \colon |x - y| \geq z \cdot \varepsilon.$

(b) $\sim(\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \forall z \in \mathbb{R} \colon |x - y| < z \cdot \varepsilon).$

$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \forall z \in \mathbb{R} \colon |x - y| < z \cdot \varepsilon).$

(c) $\forall x \in \mathbb{R},\ \sim(\exists y \in \mathbb{R},\ \sim(\forall z \in \mathbb{R} \colon |x - y| < z \cdot \varepsilon)).$

$\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, \exists z \in \mathbb{R} \colon |x - y| \geq z \cdot \varepsilon.$

3. (15 points; 5 points each) Identify each of the following statements as true or false (circle "**T**" or "**F**"). **Please explain your answers:** If a statement is true, explain why (you don't need to provide a complete proof; just a sentence or two will do). If a statement is false, provide a counterexample to the statement, and explain why it's a counterexample. (Hint: you might want to remember that every integer, including zero, divides zero, and every integer, including zero, divides itself.)

(a) $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, \forall k \in \mathbb{Z} : (m-k)|n$. **T** **F**

For example, if $m = 12$, $n = 7$, and $k = 3$, then $(m-k) \nmid n$, since $9 \nmid 7$.

(b) $\exists m \in \mathbb{Z}, \exists n \in \mathbb{Z}, \exists k \in \mathbb{Z} : (m-k)|n$. **T** **F**

For example, choose $m = 20$, $n = 5$, $k = 19$.

(c) $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z} : (m-k)|n$. **T** **F**

Given $m, n \in \mathbb{Z}$, choose $k = m - n$. Then $m - k = m - (m-n) = n$, so $(m-k)|n$.

4. (16 points) Use the principle of mathematical induction to prove the following. Please supply a complete proof; a series of unconnected calculations will not suffice.

Theorem. $\forall n \in \mathbb{N}$,

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

(Algebra hint: at a certain point, after getting a common denominator, you may want to factor out an $(n+1)^2$.)

Proof. Let $A(n)$ be the statement in question.

Step 1: Is $A(1)$ true?

$$1^3 \stackrel{?}{=} \frac{1^2 \cdot (1+1)^2}{2}$$

$$1 = 1,$$

so $A(1)$ is true.

Step 2: Assume

$$A(k) : 1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}.$$

Then

$$\begin{aligned}
1^3 + 2^3 + 3^3 + \cdots + (k+1)^3 &= 1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} = \frac{(k+1)^2(k^2+4(k+1))}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4},
\end{aligned}$$

so $A(k+1)$ follows.

So, by the principle of mathematical induction, $A(n)$ is true for all n. $\square$

5. (20 points; 5 points each) For this problem, please express all final answers in the form of a single integer, like 17 or 1,324,432,151.

Consider *lists* formed from the the ten letters A, B, C, D, E, F, G, H, I, J.

(a) How many length-4 lists can be made from these letters if repetition **is not allowed**?

$10 \cdot 9 \cdot 8 \cdot 7 = 5{,}040$ lists.

(b) How many length-4 lists can be made from these letters if repetition **is not allowed** and the list must contain *no* D's?

$9 \cdot 8 \cdot 7 \cdot 6 = 3{,}024$ lists.

(c) (Continued from the previous page.) How many length-4 lists can be made from these letters if repetition **is not allowed** and the list must contain *at least one* D?

$5{,}040 - 3{,}024 = 2{,}016$ lists.

(d) How many length-4 lists can be made from these letters if repetition **is allowed** – that is, the same letter can appear more than once – but no two *consecutive* letters can be the same?

$10 \cdot 9 \cdot 9 \cdot 9 = 7{,}290$ lists.

6. (20 points; 5 points each) For this problem, please express all final answers in the form of a single integer – like 17 or 1,324,432,151.

Consider 5-card *hands* (where order doesn't matter) formed from a standard 52-card deck of cards.

(a) How many 5-card hands contain exactly three 7's? Explain. (Caution: after you've chosen the three 7's, you still need to choose the other two cards.)

$$\binom{4}{3} \cdot \binom{48}{2} = 4{,}512$$

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(b) How many 5-card hands contain exactly two 8's? Explain.

$$\binom{4}{2} \cdot \binom{48}{3} = 103{,}776$$

.

(c) How many 5-card hands contain exactly three 7's *and* exactly two 8's? Explain.

$$\binom{4}{3} \cdot \binom{4}{2} = 24$$

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(d) How many 5-card hands contain exactly three 7's *or* exactly two 8's (or both)? Explain.

$$4{,}512 + 103{,}776 - 24 = 108{,}264.$$