EXAM 1: COMPLETELY RANDOM PRACTICE PROBLEMS

1. Using only set facts and proof strategies from your Exam 1 fact sheet, prove that, if A, B, and C are sets, and $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Solution. Let A, B, and C be sets. Assume that $A \subseteq B$ and $B \subseteq C$.

Now let $x \in A$. Since $A \subseteq B$, which is equivalent to the statement

$$x \in A \Rightarrow x \in B,$$

we conclude that $x \in B$. But then, since $B \subseteq C$, which is equivalent to the statement

$$x \in B \Rightarrow x \in C,$$

we conclude that $x \in C$.

So $x \in A \Rightarrow x \in C$. So $A \subseteq C$. $\square$

2. Prove that $(1 + 3\mathbb{Z}) \cap (1 + 2\mathbb{Z}) = 1 + 6\mathbb{Z}$. You may use the fact that a product of odd numbers is odd.

Solution.

1) First we prove that $(1 + 3\mathbb{Z}) \cap (1 + 2\mathbb{Z}) \subseteq 1 + 6\mathbb{Z}$: let $x \in (1 + 3\mathbb{Z}) \cap (1 + 2\mathbb{Z})$. Then $x \in 1 + 3\mathbb{Z}$ and $x \in 1 + 2\mathbb{Z}$, by definition of intersection. So we can write $x = 1 + 3k$ for some $k \in \mathbb{Z}$, and $x = 1 + 2m$ for some $m \in \mathbb{Z}$. Equating these two expressions for x gives $1 + 3k = 1 + 2m$, or $3k = 2m$. So $3k$ is even. But 3 is odd, so k must be even, otherwise $3k$ would be odd (since a product of odd numbers is odd). So $k = 2n$ for some $n \in \mathbb{Z}$. So $x = 1 + 3k = 1 + 3(2n) = 1 + 6n$. So $x \in 1 + 6\mathbb{Z}$.

So $(1 + 3\mathbb{Z}) \cap (1 + 2\mathbb{Z}) \subseteq 1 + 6\mathbb{Z}$.

1) Next, we prove that $1 + 6\mathbb{Z} \subseteq (1 + 3\mathbb{Z}) \cap (1 + 2\mathbb{Z})$: let $x \in 1 + 6\mathbb{Z}$. Then $x = 1 + 6n$ for some $n \in \mathbb{Z}$. But then $x = 1 + 3(2n) \in 1 + 3\mathbb{Z}$, and $x = 1 + 2(3n) \in 1 + 2\mathbb{Z}$. So, by definition of intersection, $x \in (1 + 3\mathbb{Z}) \cap (1 + 2\mathbb{Z})$.

So $1 + 6\mathbb{Z} \subseteq (1 + 3\mathbb{Z}) \cap (1 + 2\mathbb{Z})$

Therefore, $(1 + 3\mathbb{Z}) \cap (1 + 2\mathbb{Z}) = 1 + 6\mathbb{Z}$. $\square$

3. Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$. Find $\mathscr{P}(A) \cap \mathscr{P}(B)$.

Solution. We have

$$\mathscr{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

and

$$\mathscr{P}(B) = \{\emptyset, \{2\}, \{3\}, \{4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{2, 3, 4\}\},$$

so

$$\mathscr{P}(A) \cap \mathscr{P}(B) = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\}.$$

4. Let $A = \{2, \{2\}, \{2,3\}, 3\}$.

(a) Find $\mathscr{P}(A)$.

(b) Find $A \cap \mathscr{P}(A)$.

Solution. We have

$$\begin{aligned}\mathscr{P}(A) = \{&\emptyset, \\ &\{2\}, \{\{2\}\}, \{\{2,3\}\}, \{3\}, \\ &\{2,\{2\}\}, \{2,\{2,3\}\}, \{2,3\}, \{\{2\},\{2,3\}\}, \{\{2\},3\}, \{\{2,3\},3\}, \\ &\{2,\{2\},\{2,3\}\}, \{2,\{2,3\},3\}, \{2,\{2\},3\}, \{\{2\},\{2,3\},3\}, \\ &\{2,\{2\},\{2,3\},3\}\}\end{aligned}$$

so

$$\mathscr{A} \cap \mathscr{P}(A) = \{\{2\}, \{2,3\}\}.$$

(Note: for clarity, in listing the elements of $\mathscr{P}(A)$, we wrote the zero-element subsets on one line, then the one-element subsets on the next, then the two-element subsets, then three, then four. We also used red commas between elements.)

5. Given sets A, B, and C, draw a Venn diagram depicting:

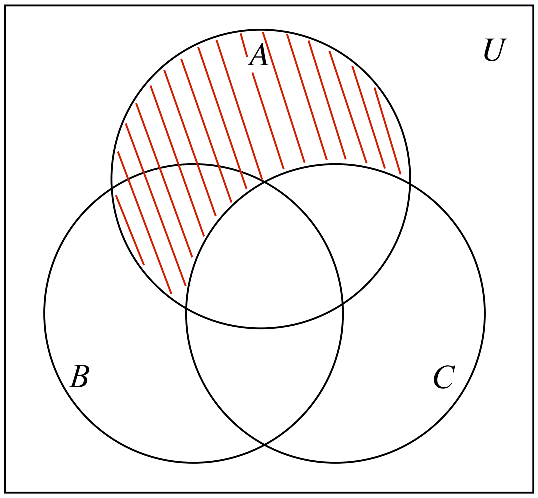
(a) $(A \cup B) \cap (A - C)$

(b) $(A \cap B) \cup (A - C)$

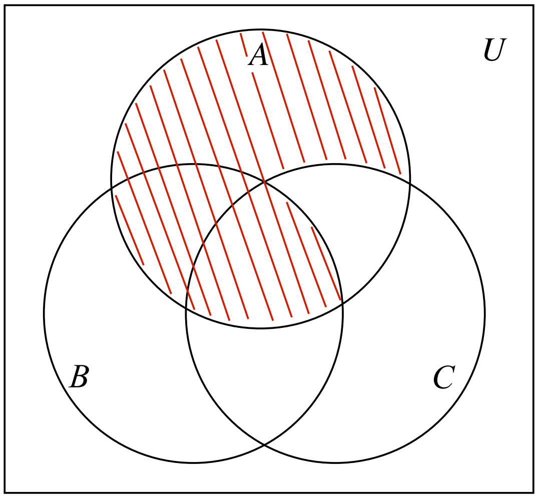
(c) $\big(B - A\big) \cup \big(C - \overline{A}\big)$

(d) $\big(B - A\big) \cap \big(C - \overline{A}\big)$

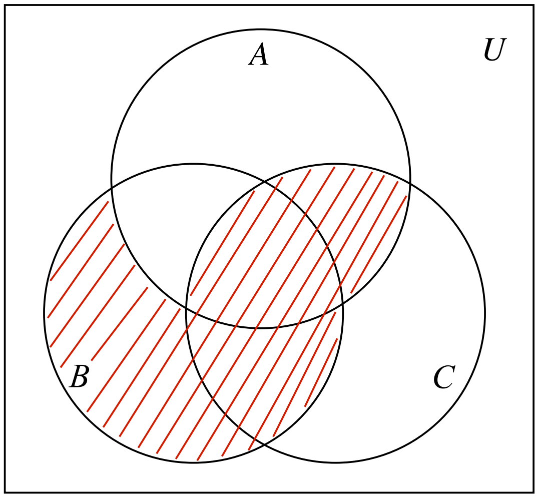
Solution.



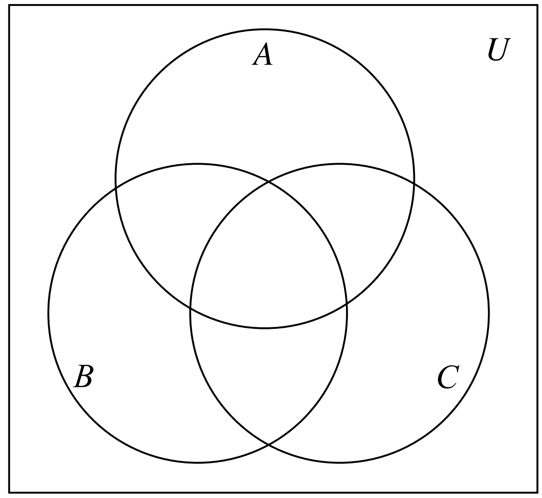
(a)



(b)



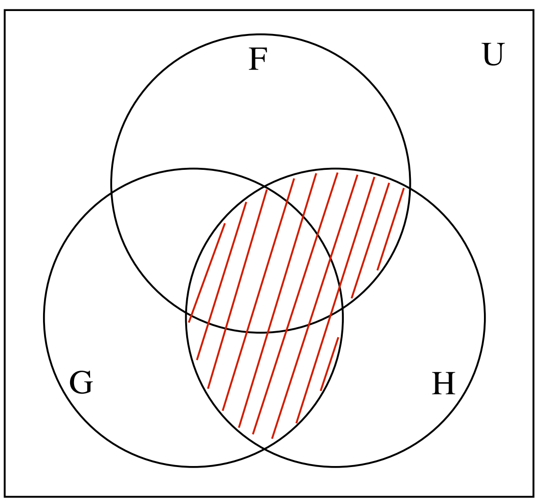
(c)



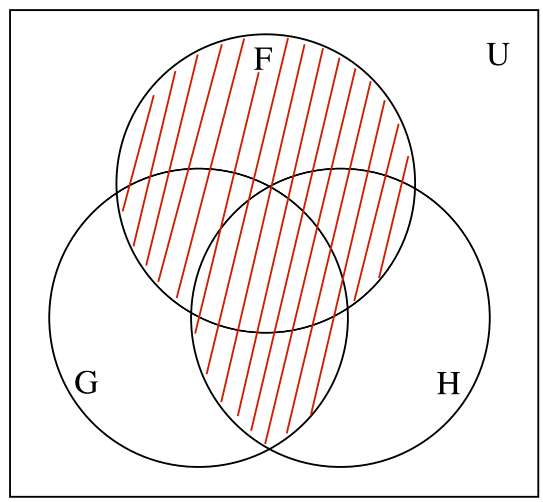
(d)

6. (a) Shade in the indicated set for each of the Venn diagrams below.

$(F \cup G) \cap H$



$F \cup (G \cap H)$



(b) Based on your shadings, what relation do you see between the set you shaded in on the left and the one you shaded in on the right? Your answer should involve the sets F, G, and H, and perhaps things like unions, subsets, intersections, etc.

(c) Prove the relation that you described in part (b) of this problem.

Solution. (a) See diagrams above.

(b) $(F \cup G) \cap H \subseteq F \cup (G \cap H)$.

(c) **Theorem.** For all sets F, G, and H,

$$(F \cup G) \cap H \subseteq F \cup (G \cap H).$$

Proof. Let F, G, and H be sets. Let $x \in (F \cup G) \cap H$. Then, by definition of intersection, $x \in F \cup G$ and $x \in H$. By definition of union, either $x \in F$ or $x \in G$. In the first case, we have $x \in F \cup (G \cap H)$, by definition of union. In the second case, since $x \in H$, we have $x \in G \cap H$, by definition of intersection, so $x \in F \cup (G \cap H)$, by definition of union. So in either case, $x \in F \cup (G \cap H)$.

Therefore, $(F \cup G) \cap H \subseteq F \cup (G \cap H)$. $\square$

7. For each $n \in \mathbb{N}$, let A_n be the open interval $\left(\frac{1}{n+1}, 2 - \frac{1}{n+1}\right)$. Find:

(a) $\displaystyle\bigcup_{n=1}^{4} A_n = \left(\frac{1}{5}, \frac{9}{5}\right)$.

(b) $\displaystyle\bigcap_{n=1}^{4} A_n = \left(\frac{1}{2}, \frac{3}{2}\right)$.

(c) $\displaystyle\bigcup_{n=1}^{\infty} A_n = (0, 2)$.

(d) $\displaystyle\bigcap_{n=1}^{\infty} A_n = \left(\frac{1}{2}, \frac{3}{2}\right)$.