MATH 2001-004: Intro to Discrete Math

October 2, 2024

First In-class Midterm Exam (SOLUTIONS)

1. (40 points; 8 points each)

Consider sets Q, R, S, and T defined by

$$Q = \{3, 6, 9, 12, 15, 18, 21, 24\}, \qquad R = \{\text{odd integers}\} = 1 + 2\mathbb{Z},$$
$$S = \{3 + 4k : -2 \leq k \leq 3\}, \qquad T = \{5, 6, 7, 8, 9, 10, 11\}.$$

Also let the universe U be the set $\mathbb{Z}$ of all integers. Find:

(a) $Q - S = \{6, 9, 12, 18, 21, 24\}$

(b) $(Q \cup T) - \overline{R} = \{3, 5, 7, 9, 11, 15, 21\}$

(c) $Q \cap T = \{6, 9\}$

(d) $(Q \cap T) \times (Q \cap T) = \{(6, 6), (6, 9), (9, 6), (9, 9)\}$

(e) $\mathscr{P}(Q \cap T) = \{\emptyset, \{6\}, \{9\}, \{6, 9\}\}$

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2. (28 points; 7 points each) (Note: for this problem, it might help to draw some pictures.) For each $n \in \mathbb{N}$, define a set A_n by

$$A_n = \left[\frac{1}{n}, 3 + n\right) = \left\{x \in \mathbb{R} : \frac{1}{n} \leq x < 3 + n\right\}.$$

Find:

(a) $\displaystyle\bigcup_{n=1}^{3} A_n = \left[\tfrac{1}{3}, 6\right)$

(b) $\displaystyle\bigcap_{n=1}^{3} A_n = [1, 4)$

(c) $\displaystyle\bigcup_{n=1}^{\infty} A_n = (0, \infty)$

(d) $\displaystyle\bigcap_{n=1}^{\infty} A_n = [1, 4)$

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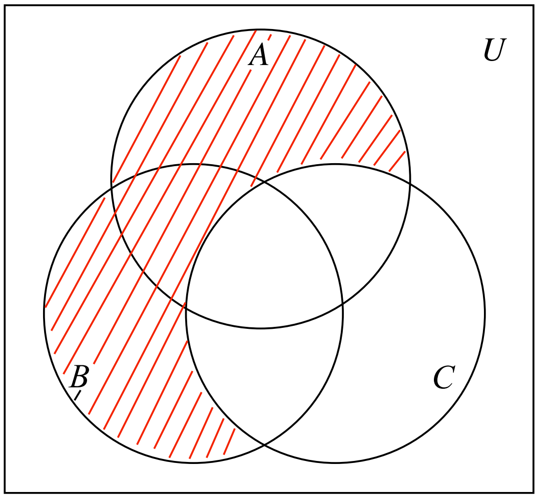
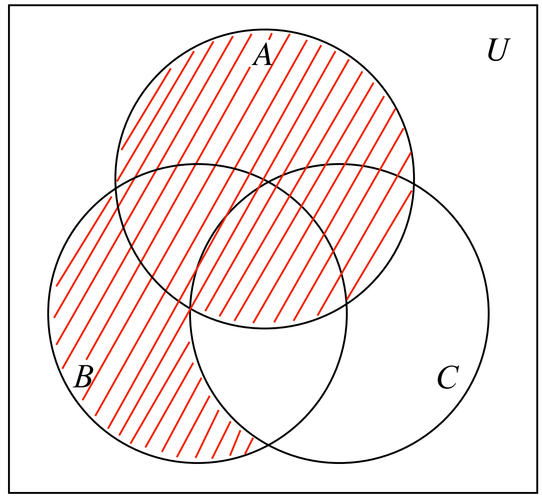
3. (12 points; 6 points each)

(a) Shade in the indicated set for each of the Venn diagrams below.

$(A \cup B) - C$



$A \cup (B - C)$



(b) Based on your shadings, which of the following relations seems to be true, for any sets A, B, and C? Circle the correct answer, (i), (ii), (iii), (iv), or (v). You don't have to prove anything. Just circle the relation that your answer to part (a) of this problem illustrates.

(i) $(A \cup B) - C \subseteq A \cup (B - C)$

(ii) $A \cup (B - C) \subseteq (A \cup B) - C$

(iii) $(B \cup C) - (C - A) \subseteq (A \cap C) - B$

(iv) $A \cup (B - C) = (A \cup B) - C$

(v) none of the above

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4. (10 points; one point for each blank) Fill in the blanks (there are 140 of them) to complete the proof of the following theorem:

Theorem. For any sets A, B, and C, we have

$$C - B \subseteq C - (B - A).$$

Proof. Let A, B, and __C__ be sets.

Let $x \in$ __$C - B$__. Then $x \in$ __C__ and $x \notin$ __B__, by definition of set difference.

Now note that, since $x \notin B$, it must be true that $x \notin B - A$, for the following reason. If $x \in B - A$, then $x \in B$, by definition of set __difference__. So, by the contrapositive, $x \notin B \Rightarrow x \notin$ __$B - A$__.

We have shown that, if $x \in C - B$, then $x \in C$ and $x \notin$ __$B - A$__. But this implies that $x \in C - (B - A)$, by definition of set __difference__. In other words,

$$\underline{\quad C - B \quad} \subseteq C - (B - A),$$

and we're done. **ATWMR**

(In the last blank above, supply an end-of-proof tagline of your own devising.)

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5. (10 points) Prove the following theorem. Use one of the given proof templates, and use only the definitions of intersection, union, and subset. Whenever you use a definition, say so. Hint: at some point, you'll want to break things into two cases. (If you use the correct proof template, and use it correctly, you'll get partial credit, even if you don't know how to do all of the "[Now do what you need to conclude:]" part.)

Theorem.

For any sets X, Y, and Z,

$$X \cap (Y \cup Z) \subseteq (X \cap Y) \cup Z.$$

Proof.

Let X, Y, and X be sets. Assume $x \in X \cap (Y \cup Z)$. Then $x \in X$ and $x \in Y \cup Z$, by definition of intersection. So, since $x \in Y \cup Z$, we have $x \in Y$ or $x \in Z$, by definition of union. We consider two cases.

(1) $x \in Y$. Then, since $x \in X$ as already noted, we have $x \in X \cap Y$, by definition of intersection. So $x \in (X \cap Y) \cup Z$, by definition of union.

(2) $x \in Z$. Then $x \in (X \cap Y) \cup Z$, by definition of union.

So in either case, $x \in (X \cap Y) \cup Z$.

Therefore, $X \cap (Y \cup Z) \subseteq (X \cap Y) \cup Z$. **ATWMR**