**Activity: Exam 2 review**

**1.** Use the principle of mathematical induction to prove that, for any $n \in \mathbb{N}$,

$$1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + 4 \cdot 6 + \cdots + n(n+2) = \frac{n(n+1)(2n+7)}{6}.$$

Please make sure to write this out carefully as a complete proof. (State what $A(n)$ is, identify your base step, induction hypothesis, conclusion, etc.)

**Proof.**

Let $A(n)$ be the identity in question.

Step 1: Is $A(1)$ true?

$$\begin{aligned} 1 \cdot 3 &\stackrel{?}{=} \frac{1(1+1)(2 \cdot 1 + 7)}{6} \\ 3 &\stackrel{?}{=} \frac{1 \cdot 2 \cdot 9}{6} \\ 3 &= 3, \end{aligned}$$

so $A(1)$ is true.

Step 2: Assume

$$A(k): \quad 1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + 4 \cdot 6 + \cdots + k(k+2) = \frac{k(k+1)(2k+7)}{6}.$$

Then

$$\begin{aligned} &1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + 4 \cdot 6 + \cdots + (k+1)(k+3) \\ &= \big(1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + 4 \cdot 6 + \cdots + k(k+2)\big) + (k+1)(k+3) \\ &= \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3) = \frac{k(k+1)(2k+7) + 6(k+1)(k+3)}{6} \\ &= \frac{(k+1)\big(k(2k+7) + 6(k+3)\big)}{6} = \frac{(k+1)\big(2k^2 + 7k + 6k + 18\big)}{6} \\ &= \frac{(k+1)\big(2k^2 + 13k + 18\big)}{6} = \frac{(k+1)\big((k+2)(2k+9)\big)}{6} \\ &= \frac{(k+1)\big(\big((k+1)+1\big)\big(2(k+1)+7\big)}{6}, \end{aligned}$$

so $A(k+1)$ follows.

So, by the principle of mathematical induction, $A(n)$ is true for all $n \in \mathbb{N}$. $\square$

**2.** Identify each of the statements as true or false, by putting a "T" or "F" in the space to the *left* of the statement. Then, in the space to the *right* of the statement, put the *number* of the statement that is the *negation* of the statement in question. For example, if the negation of statement 2 is statement 7, then put a "7" in the space to the right of statement 2.

One of the statements has no negation present, so leave the space to the right of that statement blank.

(Recall that $\mathbb{R}^+$ denotes the set of positive real numbers.)

1. F $\forall w \in \mathbb{R}^+, \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \forall z \in \mathbb{R}^+, w + z \leq x - y$ 3

2. F $\exists w \in \mathbb{R}^+, \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, \forall z \in \mathbb{R}^+, w + z \leq x - y$ 7

3. T $\exists w \in \mathbb{R}^+, \exists x \in \mathbb{R}, \exists y \in \mathbb{R}, \exists z \in \mathbb{R}^+, x < w + y + z$ 1

4. T $\sim(\sim(\forall w \in \mathbb{R}^+, \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, \exists z \in \mathbb{R}^+, x - y < w + z))$ 5

5. F $\exists w \in \mathbb{R}^+, \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \forall z \in \mathbb{R}^+, w + z \leq x - y$ 4

6. F $\forall w \in \mathbb{R}^+, \exists x \in \mathbb{R}, \forall y \in \mathbb{R}, \exists z \in \mathbb{R}^+, w + z \leq x - y$ ____

7. T $\forall w \in \mathbb{R}^+, \forall x \in \mathbb{R}, \sim(\forall y \in \mathbb{R}, \forall z \in \mathbb{R}^+, w + z \leq x - y)$ 2

8. F $\sim(\forall w \in \mathbb{R}^+, \forall x \in \mathbb{R}, \sim(\forall y \in \mathbb{R}, \forall z \in \mathbb{R}^+, x - y < w + z))$ 9

9. T $\forall w \in \mathbb{R}^+, \forall x \in \mathbb{R}, \exists y \in \mathbb{R}, \exists z \in \mathbb{R}^+, w + z \leq x - y$ 8

**3.** You wash 14 pairs of socks, each pair a different color from the other pairs, and realize, after taking everything out of the dryer, that 8 socks are lost. So you are left with $(14 \cdot 2) - 8 = 20$ socks. Find the probability that:

A. You are left with 10 matching pairs (this is the best case scenario);

B. You are left with 6 matching pairs (this is the worst case scenario).

Some hints: (i) First, what is the total number of ways of losing 8 out of 28 socks? (ii) To be left with 10 matching pairs is to say that 4 of the original 14 pairs were lost. (iii) To be left with 6 matching pairs is to say that 8 socks, all of different colors, were lost. How many ways are there of choosing the 8 colors lost, and for each of these colors, how many ways are there of choosing a sock of that color? (iv) The *probability* of scenario A is the number of ways this scenario can occur, divided by the total number of ways of losing 8 out of 28 socks. Similarly for scenario B.

There are $\binom{28}{8}$ equally likely ways of choosing eight socks from the 28 total socks. How many of these ways yield the best case scenario? Well, to be left with ten matching pairs is to say that four of the original 14 pairs were lost. There are $\binom{14}{4}$ ways of choosing four pairs from the 14 total pairs. So

$$P(\text{best case scenario}) = \frac{\binom{14}{4}}{\binom{28}{8}} \approx 0.000322061.$$

Now how many of the $\binom{28}{8}$ ways of choosing eight socks yield the worst case scenario? Well, to be left with six matching pairs is to say that eight of the original pairs were unpaired, meaning the eight socks lost consist of one sock of each color. There are $\binom{14}{8}$ ways of choosing the eight colors, and for each color chosen, there are two ways of choosing a sock of that color. So

$$P(\text{worst case scenario}) = \frac{\binom{14}{8} \cdot 2^8}{\binom{28}{8}} \approx 0.247343.$$

So $P(\text{worst case scenario}) > P(\text{best case scenario})$ (by a lot).