Fibonacci numbers.

A) Fibonacci and induction.

Consider the sequence of natural numbers that starts with

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

These are the Fibonacci numbers F_n, defined recursively by

$$F_1 = 1, \ F_2 = 1, \qquad (*)$$

$$F_{n+2} = F_{n+1} + F_n \text{ for } n \geq 1. \qquad (*')$$

Properties of Fibonacci numbers are often proved by induction, using the "initial values" $(*)$ and the "recursion formula" $(*')$.
For example:

Proposition.
$\forall n \in \mathbb{N}$ with $n \geq 2$,

$$F_{n-1} F_{n+1} - F_n^2 = (-1)^n.$$

Proof.
Let $A(n)$ be the identity claimed.

Step 1: is $A(2)$ true?

$$F_1 F_3 - F_2^2 \stackrel{?}{=} (-1)^2$$
$$1 \cdot 2 - 1^2 \stackrel{?}{=} 1$$
$$2 - 1 \stackrel{?}{=} 1$$
$$1 = 1 \ \checkmark$$

So $A(1)$ is true.

Step 2: Assume

$$A(k): F_{k-1}F_{k+1} - F_k^2 = (-1)^k$$

To deduce

$$A(k+1): F_k F_{k+2} - F_{k+1}^2 = (-1)^{k+1},$$

we note that

$$F_k F_{k+2} - F_{k+1}^2 \overset{\text{apply }(*')\text{ to } F_{k+2} \text{ and } F_{k+1}}{=} F_k(F_{k+1}+F_k) - F_{k+1}(F_k + F_{k-1})$$

$$\overset{\text{multiply out}}{=} \cancel{F_k F_{k+1}} + F_k^2 - \cancel{F_{k+1} F_k} - F_{k+1}F_{k-1}$$

$$= F_k^2 - F_{k+1}F_{k-1}$$

$$= -(F_{k-1}F_{k+1} - F_k^2)$$

$$\overset{\text{by the induction hypothesis}}{=} -(-1)^k$$

$$= (-1)^{k+1}.$$

Therefore, $A(k) \Rightarrow A(k+1)$.
So, by induction, $A(n)$ is true $\forall n \in \mathbb{N}$. $\square$

B) A formula.

COOL FACT (proof omitted: see S-POP, Proposition B(v)-1): there's a <u>closed formula</u> for the n^{th} Fibonacci number:

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \quad (n \geq 1).$$

Remark: the number $\frac{1+\sqrt{5}}{2}$ appearing above is called the <u>golden ratio</u>, denoted Φ.

c) Ratios of Fibonacci numbers.

We define a sequence R_n of <u>ratios</u> of Fibonacci numbers by

$$R_n = \frac{F_{n+1}}{F_n} \quad (n \geq 1),$$

where F_n is the n^{th} Fibonacci number.

The sequence of R_n's begins

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \ldots$$

$$\approx 1, 2, 1.5, 1.6667, 1.6, 1.625, 1.6154, 1.6191, 1.6177, \ldots$$

In fact:

<u>Theorem</u>.

$$\lim_{n\to\infty} R_n = \Phi = \frac{1+\sqrt{5}}{2} \approx 1.61803.$$

<u>Proof</u> (sketch)

Suppose $\lim_{n\to\infty} R_n = L$.

We'll prove that $L = \Phi$.

To do so, recall that

$$F_{n+2} = F_{n+1} + F_n.$$

Divide through by F_{n+1} to get

$$\frac{F_{n+2}}{F_{n+1}} = 1 + \frac{F_n}{F_{n+1}}.$$

In terms of R_n, this says

$$R_{n+1} = 1 + 1/R_n. \qquad (*)$$

Now let $n \to \infty$. Suppose R_n has the limit L. Then R_{n+1} has the same limit. So $(*)$ gives

$$L = 1 + 1/L.$$

Multiply by L:

$$L^2 = L + 1, \text{ or}$$

$$L^2 - L - 1 = 0.$$

By the quadratic formula,

$$L = \frac{1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \frac{1 \pm \sqrt{5}}{2}.$$

Now $\frac{1-\sqrt{5}}{2} < 0$, and L can't be < 0 since all of the R_n's are > 0. Conclusion:

$$L = \frac{1+\sqrt{5}}{2} = \Phi \ (\approx 1.61803).$$

The limit of ratios of consecutive Fibonacci numbers <u>is</u> the golden ratio!