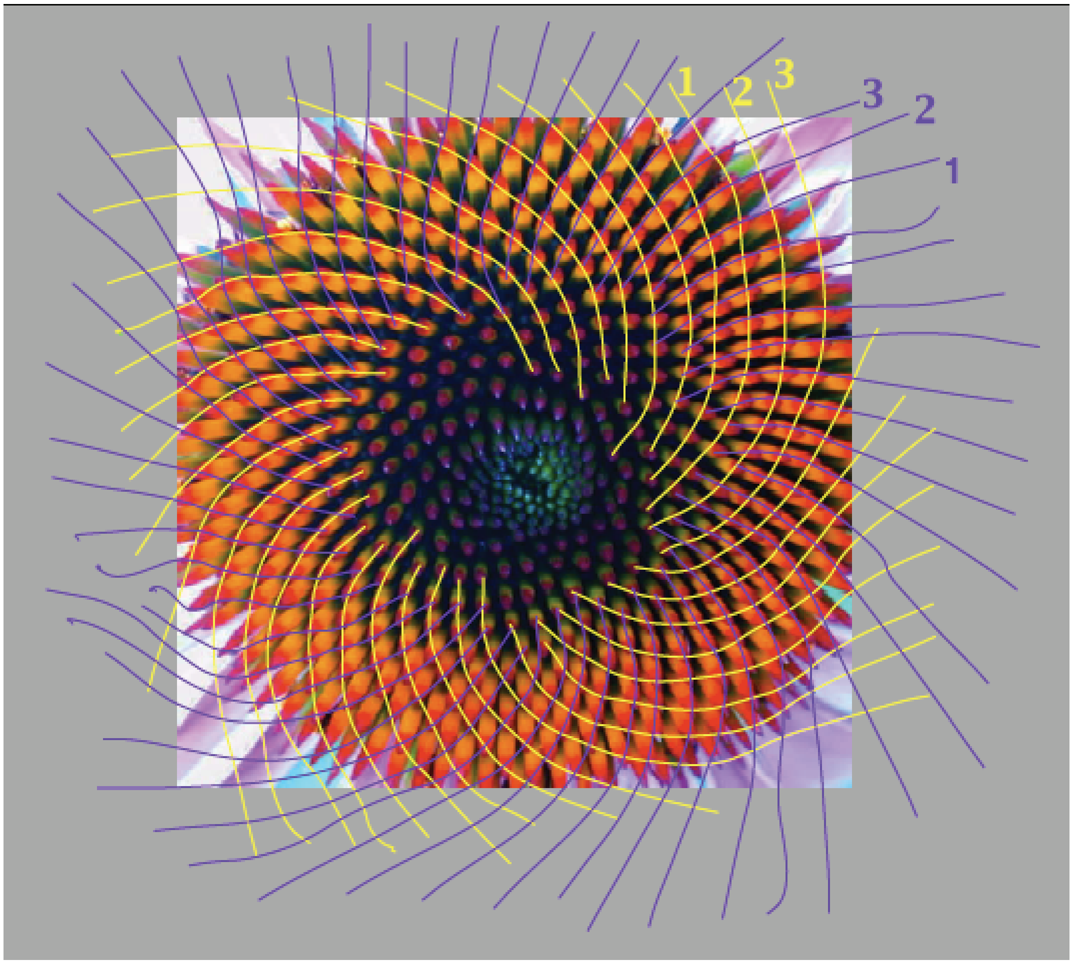
Fibonacci numbers and the golden ratio

Exercise 1. Carefully count the number of clockwise and counterclockwise spiral arms in the coneflower below.



Clockwise spiral arms: 34 Counterclockwise spiral arms: 55

Now consider the *Fibonacci sequence* F_n, which looks like this:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots.$$

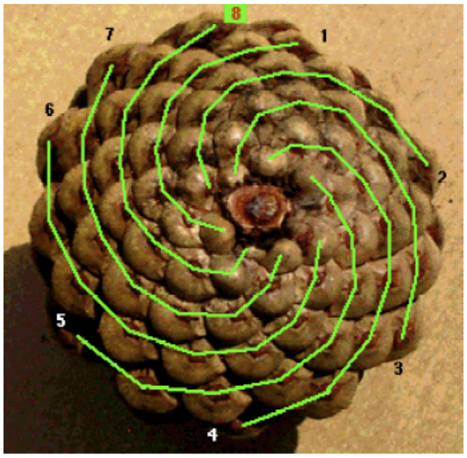
As we'll see in class, this sequence is defined by

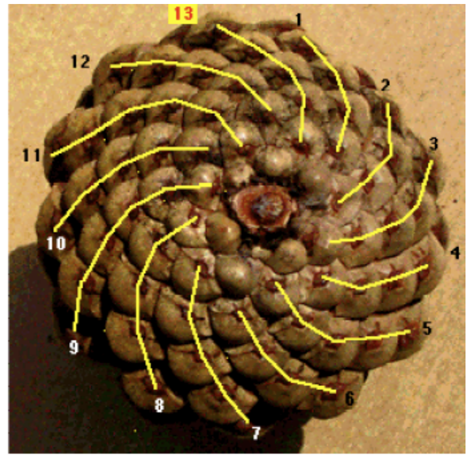
$$F_1 = F_2 = 1, \quad (*)$$

$$F_{n+2} = F_{n+1} + F_n \quad (n \geq 1). \quad (*')$$

You may have noticed that the numbers of spiral arms of the above coneflower are Fibonacci numbers!

FACT: Fibonacci numbers are everywhere! For example, count clockwise and counterclockwise spiral arms on a pine cone:





You'll get consecutive Fibonacci numbers! Similar things happen with sunflowers, pineapples, broccoli florets, etc. See

`https://en.wikipedia.org/wiki/Fibonacci_sequence`

Exercise 2. Write down the next nine Fibonacci numbers after those listed on the previous page.

34, 55, 89, 144, 233, 377, 610, 987, 1597,...

Exercise 3. Starting with $n = 2$ and continuing until at least $n = 10$ (go further if you want), compute $F_{n-1} \cdot F_{n+1} - F_n^2$. Write your answers in the space below, and then fill in the conjecture at the bottom of the page. (We'll prove this conjecture later, using induction.)

-1, 1, -1, 1, -1, 1, -1, 1, -1, 1, . . .

Conjecture: $F_{n-1} \cdot F_{n+1} - F_n^2 = \underline{\quad (-1)^n \quad}$.

Particularly interesting things happen when we examine *ratios* of successive Fibonacci numbers. Let's do this.

We define a sequence R_n by

$$R_n = \frac{F_{n+1}}{F_n}$$

for $n \geq 1$ (F_n denotes the nth Fibonacci number, as above). So the sequence R_n starts like this:

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \ldots.$$

Exercise 3. Write down the next nine R_n's as fractions. Then write these nine terms as decimal numbers, with at least four places after the decimal point. Do the R_n's appear to be converging? That is, do they appear to have a limit? If so, what (approximately) does this limit appear to be (to as many decimal places as you care to speculate)?

$$\frac{21}{13} = 1.615385, \quad \frac{34}{21} = 1.619048, \quad \frac{55}{34} = 1.617647, \quad \frac{89}{55} = 1.618182, \quad \frac{144}{89} = 1.617976,$$

$$\frac{233}{144} = 1.618056, \quad \frac{377}{233} = 1.618026, \quad \frac{610}{377} = 1.608037, \quad \frac{987}{610} = 1.618033, \ldots.$$

The R_n's seem to be bouncing up and down around some number that seems to be around 1.61803.

The number to which your above R_n's converge is, actually, a number that shows up in various other places too.

In the next problem, we investigate several of those places, using your provided rulers.

Exercise 4(a) Take out a BuffOne Card or credit card or driver's license (just one card per group is fine). Measure, in millimeters, the length of the longer side, and the length of shorter side, of you card. Record these measurements, as well as the *proportion*, meaning longer length divided by shorter length, as a decimal to a few places, of your card here:

Longer: ___86___ Shorter: ___54___ Proportion: ___1.59259___

Now repeat the above measurements for several objects from around the room. Specifically, measure:

(b) The rectangular plate surrounding the up/down projector screen controls, located on the wall near the front of the room:

Longer: ___129___ Shorter: ___84___ Proportion: ___1.53571___

(c) The rectangular screen of the projector controller, located on Dr. S.'s desk at the front of the room:

Longer: 175 Shorter: 117 Proportion: 1.49573

(d) A single-switch switchplate, like this: . Longer: 115 Shorter: 71

Proportion: 1.61972

Some things with similar proportions are: the height (chin to crown) or your head divided by the width (ear to ear) of your head; your height divided by the height of your belly button; etc. (Feel free to do some of these measurements at home if you want, but in the name of discretion, let's not do them in class.)

Exercise 5. The number $(1+\sqrt{5})/2$, often called the *golden mean* or the *golden ratio*, and often denoted by Φ, is special. It shows up in many real-life, and mathematical, situations. What are some such situations? To answer, plug this number into your calculator, and evaluate as a decimal to a few decimal places. How does what you get compare to some of the numbers above? See especially Exercises 3 and 4 above.

$$\frac{1+\sqrt{5}}{2} \approx 1.618033989.$$

Remark. There's an awful lot of debate as to whether Φ is deeply significant or not. (Perhaps the debate itself makes it significant.)

Exercise 6. Summarize some of your observations from this activity. Especially: what is the relationship between Fibonacci numbers and the golden ratio? What does either have to do with "real life?"

The limit of ratios of consecutive Fibonacci numbers appears to be the golden ratio.

The proportions (length divided by width) of many things appear to be close to the golden ratio. Further, Fibonacci numbers appear in a variety of natural contexts.