More induction.

A) A "geometric sum" formula.

Proposition.
For any real number $a \neq 1$, and for any $n \in \mathbb{N}$,

$$1 + a + a^2 + a^3 + \dots + a^n = \frac{a^{n+1} - 1}{a - 1}.$$

Proof. Let a be a real number $\neq 1$. Let $A(n)$ be the statement in question.

Step 1. Is $A(1)$ true?

$$1 + a^1 \stackrel{?}{=} \frac{a^2 - 1}{a - 1}$$

$$1 + a \stackrel{?}{=} \frac{(a+1)(a-1)}{a-1}$$

$1 + a = a + 1$ ✓ So $A(1)$ is true.

Step 2. Assume $A(k)$:

$$1 + a + a^2 + \dots + a^k = \frac{a^{k+1} - 1}{a - 1}.$$

Then

$$1 + a + a^2 + \dots + a^{k+1}$$
$$= (1 + a + a^2 + \dots + a^k) + a^{k+1}$$
$$= \frac{a^{k+1} - 1}{a - 1} + a^{k+1} = \frac{a^{k+1} - 1}{a - 1} + \frac{a^{k+1}(a-1)}{a-1}$$

$$= \frac{\cancel{a^{k+1}} - 1 + a^{k+2} - \cancel{a^{k+1}}}{a-1} = \frac{-1 + a^{k+2}}{a-1} = \frac{a^{k+2} - 1}{a-1},$$

so $A(k+1)$ follows.
So by induction, $A(n)$ is true $\forall n \in \mathbb{N}$.

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B) A different kind of example.
(Compare with Exercises 12 and 13, HW #8.)

<u>Proposition.</u>
$\forall n \in \mathbb{N}$, $24 \mid (5^{2n} - 1)$.

<u>Proof.</u>
Let $A(n)$ be the statement $24 \mid (5^{2n} - 1)$.

Step 1: Is $A(1)$ true?

Does $24 \mid (5^{2\cdot 1} - 1)$?
$24 \mid 24$, so $A(1)$ is true.

Step 2: Assume
$A(k)$: $24 \mid (5^{2k} - 1)$.

To deduce
$A(k+1)$: $24 \mid (5^{2(k+1)} - 1)$,

add and subtract the same thing, to make things look like $A(k)$

we note that

$$\begin{aligned} 5^{2(k+1)} - 1 &= 5^{2k+2} - 1 \\ &= 5^{2k+2} - 1 + 5^{2k} - 5^{2k} \\ &= 5^{2k} - 1 + 5^{2k+2} - 5^{2k} \\ &= 5^{2k} - 1 + 5^{2k}(5^2 - 1) \\ &= 5^{2k} - 1 + 24 \cdot 5^{2k}. \end{aligned}$$

Now $24 \mid (5^{2k}-1)$ by the induction hypothesis.
Moreover, $24 \mid 24$, so $24 \mid 24 \cdot 5^{2k}$ by S-POP,
Exercise B(i)-3(b). So $24 \mid (5^{2k}-1+24\cdot 5^{2k})$,
by S-POP Exercise B(i)-3(a) So $A(k+1)$ follows.

Therefore, by induction, $A(n)$ is true $\forall n \in \mathbb{N}$. $\square$

(C) A surprising fact.

<u>Theorem.</u>
All sneakers are identical.

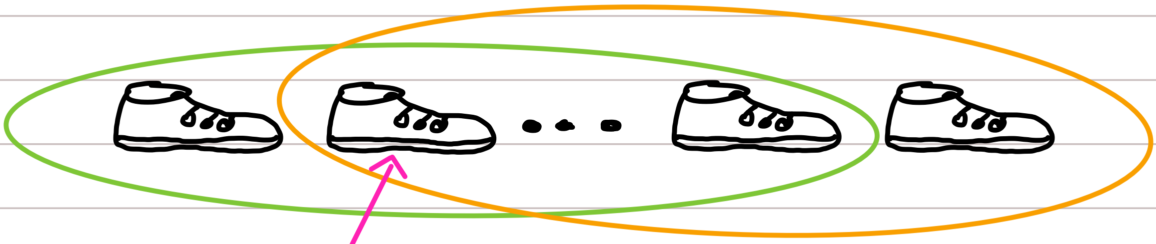
<u>Proof</u>
Let $A(n)$ be the statement:
Any n sneakers are identical.

We prove, by induction, that $A(n)$ is true $\forall n \in \mathbb{N}$.

Step 1: Is $A(1)$ true?
Any one sneaker is identical to itself, so yes, $A(1)$ is true.

Step 2: Assume
$A(k)$: any k sneakers are identical.

Now suppose we have $k+1$ sneakers.
Line them up:



(The second sneaker is part of the first group of k and the last)

By the induction hypothesis, the first k are identical, as are the last k.
So all k+1 are identical to the second one, and thus to each other.

So $A(k) \Rightarrow A(k+1)$.

So by induction, $A(n)$ is true $\forall n \in \mathbb{N}$. ATWMR

Problem with proof: The inductive step fails for $k=1$! (If $k=1$, the second sneaker is not part of both the first k and the last k.)