The Principle of Mathematical Induction.

(A proof template for statements of the form
$$\forall n \in \mathbb{N} : A(n).)$$

Idea: let $A(n)$ be a statement about a natural number n.

Example:
$$1+2+3+\dots+n = \frac{n(n+1)}{2}.$$

Suppose we can show that:

(1) $A(1)$ is true, and

(2) whenever $A(k)$ is true for $k \in \mathbb{N}$, $A(k+1)$ follows. In other words, $\forall k \in \mathbb{N}$, $A(k) \Rightarrow A(k+1)$.

Then by (1), $A(1)$ holds, so by (2), $A(2)$ holds, so by (2), $A(3)$ holds, so by (2), $A(4)$ holds... so "ultimately," $A(n)$ holds for any $n \in \mathbb{N}$.

So we have a "mathematical induction" proof template:

Theorem. $\forall n \in \mathbb{N}, A(n)$.

Proof

Step 1: Is $A(1)$ true? [Prove $A(1)$.]
So $A(1)$ is true.

Step 2: Assume $A(k)$. [Now do what's

necessary to conclude:] So $A(k+1)$ follows.

(optional)

So by (the principle of mathematical) induction, $A(n)$ is true $\forall n \in \mathbb{N}$.

$\square$

Remarks

Step 1 is the "base step".

Assuming $A(k)$ is the "induction hypothesis."

Deducing $A(k+1)$ is the "inductive step".

Example 1:

Prove that, $\forall n \in \mathbb{N}$,

$$1+2+3+\dots+n = \frac{n(n+1)}{2}.$$

Proof.

Let $A(n)$ be the statement

$$1+2+3+\dots n = \frac{n(n+1)}{2}.$$

Step 1: is $A(1)$ true?

$$1 \overset{?}{=} \frac{1(1+1)}{2}$$

$$1 = 1 \checkmark$$

So $A(1)$ is true.

Step 2: Assume

$$A(k): 1+2+3+\dots+k = \frac{k(k+1)}{2}.$$

To deduce $A(k+1)$, we note that

$$\overbrace{1+2+3+\dots+k+1}^{\text{write out the left side of } A(k+1)} = \overbrace{(1+2+3+\dots+k)+k+1}^{\text{express it in terms of the left side of } A(k)}$$

invoke (use) the induction hypothesis

$$= \frac{k(k+1)}{2} + k+1$$

do stuff to get to the right side of $A(k+1)$.

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

So $A(k+1)$ follows.

So by induction,

$$1+2+3+\dots+n = \frac{n(n+1)}{2} \quad \forall n \in \mathbb{N}. \qquad \square$$

Tips for the inductive step (in many cases):

- write out the left side of $A(k+1)$ *
- Express this in terms of the left side of $A(k)$
- Invoke the induction hypothesis.
- Turn the result into the right side of $A(k+1)$.

* in scratch somewhere, maybe write out the right side as well, so you know what to aim for

Example 2.

Prove that, $\forall n \in \mathbb{N}$,

$$\sum_{i=1}^{n} (2i-1) = n^2$$

(the sum of the first n odd natural numbers is n^2).

Proof.

Let $A(n)$ be the statement

$$\sum_{i=1}^{n} (2i-1) = n^2.$$

Step 1: is $A(1)$ true?

$$\sum_{i=1}^{1} (2i-1) = 2\cdot 1 - 1 = 1 = 1^2,$$

so $A(1)$ is true.

Step 2: Assume

$$A(k): \sum_{i=1}^{k} (2i-1) = k^2.$$

Then

$$\sum_{i=1}^{k+1} (2i-1) = \sum_{i=1}^{k} (2i-1) + 2(k+1) - 1$$

by induction hypothesis

$$= k^2 + 2(k+1) - 1 = k^2 + 2k + 1$$
$$= (k+1)^2,$$

so $A(k+1)$ follows.

So by induction, $A(n)$ is true $\forall n \in \mathbb{N}$.

ATWMR