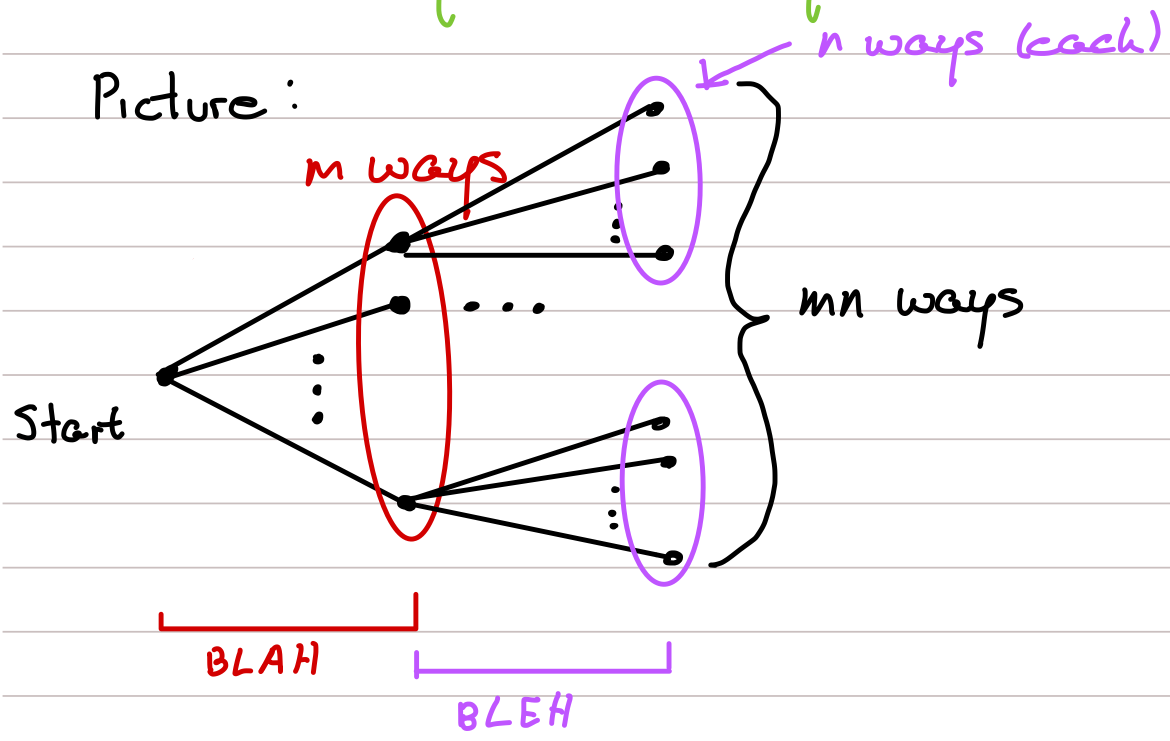
Combinatorics (a fancy word for counting).

The key is the Multiplication Principle (MP), which says:

> If there are m ways of doing BLAH and, for each of these ways, there are n ways of doing BLEH, then there are mn ways of doing BLAH followed by, or together with, BLEH.

Multiplication Principle (MP)

Picture:



We can use MP for two types of counting: (1) lists; (2) sets.

Today: (1) lists.

A list is an ordered sequence of items.

Example 1

How many six-symbol strings can be made from the letters A-Z and the digits

0-9, if:
(i) repetition is allowed;
(ii) repetition is not allowed;
(iii) repetition is not allowed, and the string must be 3 digits followed by 3 letters.

Solution

(i) $\exists\, 26 + 10 = 36$ choices for each symbol, yielding

$$36 \cdot 36 \cdot 36 \cdot 36 \cdot 36 \cdot 36 = 36^6$$
$= 2{,}176{,}782{,}336$ strings.

(ii) $36 \cdot 35 \cdot 34 \cdot 33 \cdot 32 \cdot 31 = 1{,}402{,}410{,}240$ strings
(36 choices for first symbol, leaving only 35 choices for the second, etc.)

(iii) $10 \cdot 9 \cdot 8 \cdot 26 \cdot 25 \cdot 24 = 11{,}232{,}000$ strings.

Note that, generally, MP implies:

> The number of k-element lists that can be made from a set with n elements is:
> (i) n^k allowing repetition;
> (ii) $n(n-1)(n-2)\cdots(n-k+1)$* disallowing repetition.

*Sometimes denoted $_nP_k$.

Some counting tips & tricks:

(a) If necessary, break things up into cases (then add the counts from the separate cases).

(b) To count how many lists have a property P, it's sometimes easier to compute
$|\{\text{all lists}\}| - |\{\text{lists without property } P\}|$.

(c) Be careful to subtract anything counted twice.

Example 2.

How many length-3 strings, w/o repetition, can be made from the letters A, B, C, D, E, if:

(a) the first or the second letter is an E;

(b) One letter is an E;

(c) The 2^{nd} letter is a C or the 3^{rd} is an E (or both).

Solution

(a)

$$\begin{array}{r} 1\cdot 4\cdot 3 \\ +\ \ 4\cdot 1\cdot 3 \\ \hline 24 \end{array}$$ strings

← first letter is an E

← second letter is an E

(b) $\underbrace{5\cdot 4\cdot 3}_{\text{all strings}} - \underbrace{4\cdot 3\cdot 2}_{\text{strings with no E}} = 36$ strings

(or we can add cases: 12 strings have E as first letter; 12 have E as first; 12 have E as third).

(c) $\underbrace{4\cdot 1\cdot 3}_{2^{nd}\text{ letter is a C}} + \underbrace{4\cdot 3\cdot 1}_{3^{rd}\text{ letter is an E}} - \underbrace{3\cdot 1\cdot 1}_{} = 21$ strings

we counted strings ending in CE twice, so subtract once to compensate.