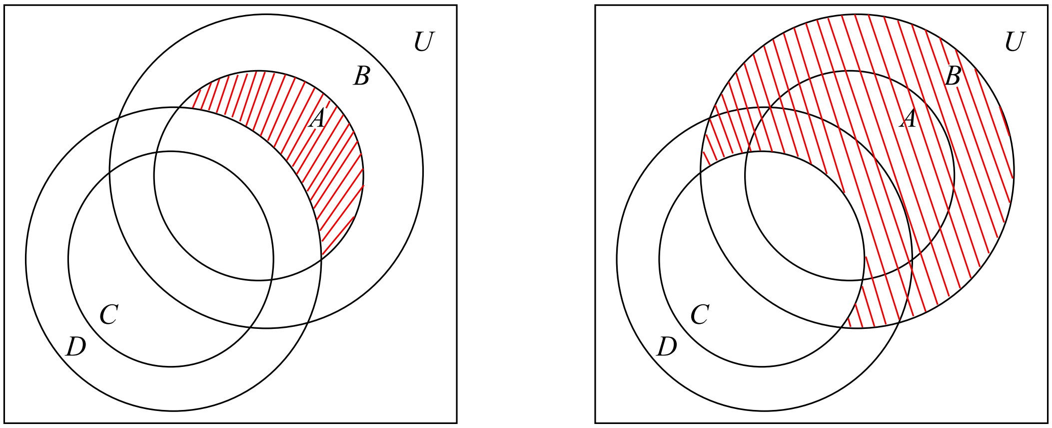
1. Consider the following pair of identical Venn diagrams, each depicting sets A, B, C, and D, where $A \subseteq B$ and $C \subseteq D$.



(a) In the diagram on the left, shade in the set $A - D$.

(b) In the diagram on the right, shade in the set $B - C$.

(c) Fill in the blank to complete the following theorem, illustrated by the above Venn diagram:

Theorem. For any sets A, B, C, and D, we have

$$A \subseteq B \text{ and } C \subseteq D \Rightarrow A - D \subseteq \underline{\quad B - C \quad}.$$

(d) Fill in the blanks to complete the proof of the above theorem.

Proof. Let A, B, C, and D be __sets__.

Assume $A \subseteq$ __B__ and $C \subseteq$ __D__. We wish to conclude that __$A - D$__ $\subseteq B - C$. To do this, assume $x \in$ __$A - D$__. Then $x \in A$ and $x \notin D$, by definition of set __difference__.

Since $x \in A$, we have $x \in$ __B__, by definition of subset.

Now the assumption $C \subseteq D$ is equivalent to the statement $x \in C \Rightarrow$ __$x \in D$__, which is equivalent to the contrapositive statement $x \notin D \Rightarrow$ __$x \notin C$__. So, since $x \notin D$ as already noted, we conclude that __$x \notin C$__. Therefore, since $x \in B$ as already noted, we have $x \in$ __$B - C$__, by definition of __set difference__.

We have shown that, if $A \subseteq B$ and $C \subseteq$ __D__, then $x \in A - D \Rightarrow$ __$x \in B - C$__. In other words,

$$A \subseteq B \text{ and } C \subseteq D \Rightarrow \underline{\quad A - D \subseteq B - C \quad},$$

and we're done. **ATWMR**

(In the last blank above, supply an end-of-proof tagline devised by your group.)

2. Let A, B, C, and D be sets. Show that the converse to the above theorem is false. That is, show (by counterexample) that it's *not* necessarily true that

$$A - D \subseteq B - C \Rightarrow A \subseteq B \text{ and } C \subseteq D.$$

Hint: You might want to look at the Venn diagrams above and think about how you could tweak them so that, while it's still true that $A - D \subseteq B - C$, it's no longer true that $A \subseteq B$ and $C \subseteq D$ both still hold. But remember that tweaking the diagrams is not enough, you still need to supply an explicit counterexample.

SOLUTION. For example, let

$$A = \{1, 2, 3, 4, 5\}, \quad B = \{1, 2, 3, 4, 5, 6, 7\}, \quad C = \{1, 2, 7, 10, \pi, 43\}, \quad D = \{1, 2, 10, 17, 53\}.$$

Then

$$A - D = \{3, 4, 5\} \quad \text{and} \quad B - C = \{3, 4, 5, 6\},$$

so $A - D \subseteq B - C$. But, while it is true that $A \subseteq B$, it's not true that $C \subseteq D$.