Quantifiers.

The symbols $\exists$ and $\forall$ are called quantifiers.

(1) $\exists$ is the existential quantifier; it means "there exists" or "for some" or "for at least one."

Also, if $Q(x)$ is a statement about an object x, and A is a set, then

$$\exists x \in A : Q(x)$$

means $Q(x)$ holds for at least one element of A.

Examples:

$\exists x \in \mathbb{N} : x \geq 42$ is true.

$\exists x \in \mathbb{N} : x < 0$ is false.

(2) $\forall$ is the universal quantifier; it means "for all" or "for every" or "for any."

Also, if $Q(x)$ is a statement regarding an object x, and A is a set, then

$$\forall x \in A : Q(x)$$

means $Q(x)$ holds for any $x \in A$.

Examples:

$\forall x \in \mathbb{N} : x \geq 42$ is false.

$\forall x \in \mathbb{N} : x \geq 0$ is true.

(3) (a) We can string quantifiers together.

Examples:

$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}: x > y$ is <u>true</u>.
$\exists x \in \mathbb{R}, \forall y \in \mathbb{R}: x > y$ is <u>false</u>.
(note that order matters!)

(b) We can negate statements with quantifiers. In particular,

$\sim(\exists x \in A : Q(x))$

is equivalent to

$\forall x \in A : \sim Q(x)$,

and

$\sim(\forall x \in A : Q(x))$

is equivalent to

$\exists x \in A : \sim Q(x)$.

(4) Examples: True or False, and why?

(a) $\sim(\forall n \in \mathbb{N},\ n^2 = n)$
(b) $\sim(\exists n \in \mathbb{N}\ \ n^2 < n)$
(c) $\forall y \in \mathbb{N}, \exists x \in \mathbb{N}: y > x$
(d) $\exists y \in \mathbb{N}, \forall x \in \mathbb{N}: y \leq x$
(e) $\sim(\exists y \in \mathbb{R}, \exists x \in \mathbb{R}: |x-y| < 0)$.
Also, rewrite this statement without using $\sim$.
(f) $\sim(\forall x \in \mathbb{R} - \mathbb{R}^+, \exists y \in \mathbb{R}^+, \exists z \in \mathbb{R}^+: x \geq yz)$.
(Recall: $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$.)

<u>SOLUTION.</u>

(a) This statement is equivalent to

$\exists n \in \mathbb{N} : n^2 \neq n$

which is true (take $n = 2$).

(b) Equivalent to
$\forall n \in \mathbb{N},\ n^2 \geq n,$
which is true. (Proof: if $n \in \mathbb{N}$, then $n \geq 1$. Multiply by n to get $n^2 \geq n$.)

(c) False. If $y = 1$, there's no $x \in \mathbb{N}$ with $y > x$.

(d) True. Let $y = 1$: then $y \leq x$ for every $x \in \mathbb{N}$.

(e) $\sim(\exists y \in \mathbb{R}, \exists x \in \mathbb{R} : |x - y| < 0)$
is equivalent to
$\forall y \in \mathbb{R}, \sim(\exists x \in \mathbb{R} : |x - y| < 0)$
which is equivalent to
$\forall y \in \mathbb{R}, \forall x \in \mathbb{R},\ |x - y| \geq 0,$
which is true (since $|z| \geq 0$ for all real numbers z).

(f) $\exists x \in \mathbb{R} - \mathbb{R}^+,\ \forall y \in \mathbb{R}^+,\ \forall z \in \mathbb{R}^+ : x < yz.$
True. Let $x = 0$. Then, for any $y, z \in \mathbb{R}^+$, $yz > 0$, so $x < yz$.