Theorem. For any sets A, B, and C, we have

$$(A \cup B) - C = (A - C) \cup (B - C).$$

Proof. Let A, B, and C be sets. To prove our theorem, we will show that

$(A \cup B) - C \subseteq (A - C) \cup (B - C)$ and $(A - C) \cup (B - C) \subseteq (A \cup B) - C$.

(1) To show that $(A \cup B) - C \subseteq (A - C) \cup (B - C)$, let $x \in (A \cup B) - C$.

Then $x \in A \cup B$ and $x \notin C$. Since $x \in A \cup B$, we have $x \in A$ or $x \in B$. We consider these two cases separately.

(a) Suppose $x \in A$. Then, since $x \notin C$, we have $x \in A - C$. Then certainly $x \in A - C$ or $x \in B - C$. So, by definition of union, $x \in (A - C) \cup (B - C)$.

(b) Suppose $x \in B$. Then, since $x \notin C$, we have $x \in B - C$. Then certainly $x \in A - C$ or $x \in B - C$. So, by definition of union, $x \in (A - C) \cup (B - C)$.

In either case, we have $x \in (A - C) \cup (B - C)$. So

$$(A \cup B) - C \subseteq (A - C) \cup (B - C).$$

(2) To show that $(A - C) \cup (B - C) \subseteq (A \cup B) - C$, let $x \in (A - C) \cup (B - C)$.

Then $x \in A - C$ or $x \in B - C$. We consider these two cases separately.

(a) Suppose $x \in A - C$. Then $x \in A$ and $x \notin C$. Since $x \in A$, certainly $x \in A$ or $x \in B$, so $x \in A \cup B$, by definition of union.

But then, since $x \notin C$, we have $x \in (A \cup B) - C$.

(b) Suppose $x \in B - C$. Then $x \in B$ and $x \notin C$. Since $x \in B$, certainly $x \in A$ or $x \in B$, so $x \in A \cup B$, by definition of union.

But then, since $x \notin C$, we have $x \in (A \cup B) - C$.

In either case, we have $x \in (A \cup B) - C$. So

$$(A - C) \cup (B - C) \subseteq (A \cup B) - C.$$

Since

$(A \cup B) - C \subseteq (A - C) \cup (B - C)$ and $(A - C) \cup (B - C) \subseteq (A \cup B) - C$,

it follows that

$$(A \cup B) - C = (A - C) \cup (B - C). \qquad \square$$