More proof strategies/templates.

Notation: if P is any statement, then $\sim P$ denotes the negation, called "not P," of P. (Sometimes, $\sim P$ is written $\neg P$.)

1) Contrapositive.

Fact. $P \Rightarrow Q$ is logically equivalent to $\sim Q \Rightarrow \sim P$. (These statements are true or false together.)

Think about it: if P necessitates Q, then for Q to fail, P must fail too.

Here's a contrapositive proof template:

Theorem. $P \Rightarrow Q$.

Proof.
Assume $\sim Q$.
[Now do stuff to get:] Therefore, $\sim P$.

So $P \Rightarrow Q$. $\square$

Example.
Let $m \in \mathbb{Z}$. Prove that, if m^2 is odd, then m is odd.

Solution.
Let $m \in \mathbb{Z}$. Suppose m is not odd.

Then m is even. So $m = 2k$ for some $k \in \mathbb{Z}$.
But then

$$\begin{aligned} m^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2) \\ &= 2n, \end{aligned}$$

where $n = 2k^2 \in \mathbb{Z}$. So m^2 is even. So m^2 is not odd.

So if m^2 is odd, then m is odd. $\square$

2) <u>$P \Leftrightarrow Q$.</u>

By definition, $P \Leftrightarrow Q$ (also written P iff Q, and read "P if and only if Q," or "P is necessary and sufficient for Q") means $P \Rightarrow Q$ and $Q \Rightarrow P$.
$P \Leftrightarrow Q$ proof template:

<u>Theorem</u>. $P \Leftrightarrow Q$.
<u>Proof</u>.
1) Assume P. [Then do stuff to show:] Therefore, Q.
2) Next, assume Q. [Then do stuff to show:]: Therefore, P.

So $P \Leftrightarrow Q$. $\square$

Remark: alternatively, in a $P \Leftrightarrow Q$ proof, you can prove $P \Rightarrow Q$ or* $Q \Rightarrow P$ by the <u>contrapositive</u>.
* Inclusive or.

Example:

<u>Theorem.</u> Let $n \in \mathbb{Z}$. Then $\overbrace{n \text{ is even}}^{P}$ iff $\underbrace{n^2+6n+5 \text{ is odd}}_{Q}$.

<u>Proof.</u>

Let $n \in \mathbb{Z}$.

1) Assume n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$. So

$$\begin{aligned} n^2+5n+6 &= (n+1)(n+5) \\ &= (2k+1)(2k+5) \\ &= (2k+1)(2(k+2)+1) \\ &= (2k+1)(2m+1), \end{aligned}$$

where $k, m \in \mathbb{Z}$. So n^2+5n+6 is a product of two odd numbers, and is therefore odd, by Exercise B(i)-1b in S-POP.[1]

2) Next, assume n is <u>not</u> even. Then n is odd, so $n = 2k+1$ for some $k \in \mathbb{Z}$. So

$$\begin{aligned} n^2+6n+5 &= (n+1)(n+5) \\ &= (2k+1+1)(2k+1+5) \\ &= (2k+2)(2k+6) \\ &= (2(k+1))(2(k+3)) \\ &= (2m)(2\ell), \end{aligned}$$

where $m, \ell \in \mathbb{Z}$. So n^2+6n+5 is a product of even numbers, and is therefore even and is not odd, by Exercise B(i)-1a in S-POP.[1]

So n is even iff n^2+6n+5 is odd.

$\square$[2]

Footnotes:

1. You can use previous results if you <u>cite</u> them.

2. Note the strategy: to prove $P \iff Q$, we proved that

   a) $P \Rightarrow Q$;
   b) $\sim P \Rightarrow \sim Q$.