$A \subseteq B$ proofs.

Recall: to say that $A \subseteq B$ is to say that, if $x \in A$, then $x \in B$. That is: $A \subseteq B$ is equivalent to

$$x \in A \Rightarrow x \in B.$$

That is, an "$A \subseteq B$" statement is a kind of "$P \Rightarrow Q$" statement.

$A \subseteq B$ proof template:

Theorem. $A \subseteq B$.

Proof.
Assume $x \in A$.
[Then do what works to show that:]
Therefore, $x \in B$.
So $A \subseteq B$. $\square$

Examples.

Theorem 1.

$$3 + 12\mathbb{Z} \subseteq 3 + 6\mathbb{Z}.$$

Proof.
Let $x \in 3 + 12\mathbb{Z}$. Then $x = 3 + 12k$ for some $k \in \mathbb{Z}$. But $12 = 6 \cdot 2$, so

$$\begin{aligned} x &= 3 + (6 \cdot 2)k \\ &= 3 + 6 \cdot (2k) \\ &= 3 + 6m, \end{aligned}$$

where $m = 2k \in \mathbb{Z}$. So $x \in 3 + 6\mathbb{Z}$.
So $3 + 12\mathbb{Z} \subseteq 3 + 6\mathbb{Z}$. $\square$

Theorem 2.

For any sets A, B, and C,

$$A \cap B \subseteq (A \cup C) \cap B.$$

Proof.

Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. But since $x \in A$, certainly $x \in A$ or $x \in C$, so $x \in A \cup C$, by definition of union.
So $x \in A \cup C$ and $x \in B$. But then, by definition of intersection, $x \in (A \cup C) \cap B$.

So $A \cap B \subseteq (A \cup C) \cap B$. $\square$

Theorem 3.

For any sets X, Y, and Z, if $X \subseteq Z$ and $Y \subseteq Z$, then $X \cup Y \subseteq Z$.

Proof.

Let X, Y, and Z be sets; assume $X \subseteq Z$ and $Y \subseteq Z$.

Now assume $x \in X \cup Y$. Then $x \in X$ or $x \in Y$. We consider these cases separately:

1. If $x \in X$ then, since $X \subseteq Z$, we have $x \in Z$.
2. If $x \in Y$ then, since $Y \subseteq Z$, we have $x \in Z$.

So in all cases, $x \in Z$. So $X \cup Y \subseteq Z$. $\square$

On Friday, we showed that, for any sets A and B,

$$A - B \subseteq (A \cup B) - (A \cap B).$$

Switching the roles of A and B, we can conclude that

$$B - A \subseteq (A \cup B) - (A \cap B).$$

From Theorem 3, it then follows that

<u>Theorem 4.</u>
For any sets A and B,

$$(A-B) \cup (B-A) \subseteq (A \cup B) - (A \cap B).$$

COOL FACT: the "$\subseteq$" in Theorem 4 may be reversed:

<u>Theorem 5.</u> For any sets A and B,
$$(A \cup B) - (A \cap B) \subseteq (A-B) \cup (B-A).$$

Proof: later.

Now, two sets X, Y are said to be <u>equal</u> if $X \subseteq Y$ and $Y \subseteq X$. So, by Theorems 4 and 5 together, we can conclude:

<u>Theorem 6.</u>
For any sets A and B,

$$(A-B) \cup (B-A) = (A \cup B) - (A \cap B).$$