An imperfect proof, and a corrected version.

A) Imperfect.

Proposition.

If $a, b \in \mathbb{Z}$ are odd, then so is ab.

$a = 2m+1 \qquad b = 2n+1$

$$ab = (2m+1)(2n+1)$$
$$4mn + 2m + 2n + 1$$
$$2(2mn + m + n) + 1$$
$$2k+1$$

So ab is odd.

B) Correction.

Proposition. (Corrections in magenta.)

If $a, b \in \mathbb{Z}$ are odd, then so is ab.

Proof.

Suppose $a, b \in \mathbb{Z}$ are odd. Then we can write

$a = 2m+1, \qquad b = 2n+1$, where $m, n \in \mathbb{Z}$.

So

$$\begin{aligned} ab &= (2m+1)(2n+1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1 \\ &= 2k+1, \text{ where } k \in \mathbb{Z}. \end{aligned}$$

optional

So ab is odd.

Therefore, if $a, b \in \mathbb{Z}$ are odd, then so is ab.

$\square$

In this activity, we interpret "$A \subseteq B$" proofs as "$P \Rightarrow Q$" proofs, and use this idea to prove some things about sets.

Fill in all of the blanks in this worksheet. (There are 24 of them, in addition to the "QUESTION" near the bottom of the next page.)

Recall that the statement "$A \subseteq B$" means: whenever x is in A, then x is also in ___B___. In other words, $A \subseteq B$ means: if $x \in A$, then $x \in$ ___B___. So the statement "$A \subseteq B$" is actually of the form $P \Rightarrow Q$, where P is the statement "$x \in A$" and Q is the statement "___$x \in B$___."

So: to PROVE a statement of the form "$A \subseteq B$," we do what we usually do in $P \Rightarrow Q$ proofs: We assume P (in this case, we assume that $x \in A$), and then do what's necessary to deduce Q (in this case, to deduce that $x \in$ ___B___).

So here's an $A \subseteq B$ **proof template:**

Theorem. $A \subseteq B$.

Proof. Assume ___$x \in A$___. [Then do what works to conclude:] Therefore, $x \in B$.

So $A \subseteq B$. **ATWMR**

Recall that "**ATWMR**," which stands for "And There Was Much Rejoicing," is a kind of goofy way of saying "The proof is done." So "**ATWMR**" is more or less equivalent to "QED" or a "□." Feel free to use your own end-of-proof tagline, but please, nothing inappropriate!

Complete the following example (which is conceptually pretty straightforward, but a good way to get familiar with this proof strategy):

Theorem. For any sets A, B, and C, we have $A \cap B \subseteq A \cup C$.

Proof. Assume that A, B, and C are sets, and that $x \in A \cap B$. Then, by definition of ___intersection___, we have $x \in A$ and $x \in B$. So in particular, $x \in A$. But then certainly $x \in A$ or $x \in C$, so by definition of union, we have $x \in$ ___A___ $\cup$ ___C___.

So $A \cap B \subseteq A \cup C$. ___**ATWMR**___

(In the last blank above, supply an end-of-proof tagline devised by your group.)

Let's do another proof.

(continued on the next page)

Theorem. For any sets A and B, we have $A - B \subseteq (A \cup B) - (A \cap B)$.

[Remark: We haven't done Venn diagrams yet (we will shortly). But if you're familiar with the notion of a Venn diagram, you might want to draw one to help illustrate this theorem.]

Proof. Let $x \in$ ___$A - B$___. To deduce that $x \in (A \cup B) - (A \cap B)$, we need to demonstrate two things: first, that $x \in A \cup B$, and second, that $x \notin$ ___$A \cap B$___. We do so as follows:

1. First, we show $x \in A \cup B$. Since $x \in A - B$ by assumption, we have $x \in A$ and x ___$\notin$___ B. In particular, $x \in$ ___A___. It follows that $x \in A$ or $x \in B$, so by definition of ___union___, $x \in A \cup B$.
2. Second, we show $x \notin$ ___$A \cap B$___. Since $x \in A - B$ by assumption, we have $x \in A$ and $x \notin B$. In particular, $x \notin$ ___B___. But if $x \notin B$, then certainly x is not in both A and B, so $x \notin A$ ___$\cap$___ B.

To summarize, we've shown that, if $x \in$ ___$A - B$___, then $x \in (A \cup B) - (A \cap B)$. So ___$A - B$___ $\subseteq$ ___$(A \cup B) - (A \cap B)$___. **ATWWMR**

QUESTION: without actually writing down a proof, how would you argue that, from the above theorem, we can also deduce that $B - A \subseteq (A \cup B) - (A \cap B)$? Answer with a sentence or two in the space below. Hint: it's not hard to show (and you don't have to show) that $A \cup B = B \cup A$ and $A \cap B = B \cap A$.

We proved that, for sets A and B, $A - B \subseteq (A \cup B) - (A \cap B)$. Switching the names A and B (they're just names), we get

$$B - A \subseteq (B \cup A) - (B \cap A). \qquad (*)$$

But, as just noted, $B \cup A = A \cup B$ and $B \cap A = A \cap B$, so **(∗)** gives

$$B - A \subseteq (A \cup B) - (A \cap B).$$

Fill in these last three blanks: from the facts that $A - B \subseteq (A \cup B) - (A \cap B)$ and $B - A \subseteq (A \cup B) - (A \cap B)$, and from the definition of union, we can conclude that

$$\underline{\quad (A - B) \quad} \cup (B - A) \subseteq \underline{\quad (A \cup B) \quad} - \underline{\quad (A \cap B) \quad}.$$

In fact, the symbol "$\subseteq$" here can be replaced by "=." We'll discuss this further soon.