More on sets.

Let A, B be sets.

Definitions 1-4.

We define:

1) The union $A \cup B$ ("A union B") of A and B by

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

(in math, "or" is inclusive: it means one or the other, or both).

2) The intersection $A \cap B$ ("A intersect B") of A and B by

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$
$$= \{x \in A : x \in B\} = \{x \in B : x \in A\}.$$

3) The difference $A - B$ ("A minus B") of A and B by

$$A - B = \{x \in A : x \notin B\}.$$

4) The Cartesian product $A \times B$ ("A cross B") of A and B by

$$A \times B = \{\text{ordered pairs } (x, y) : x \in A, y \in B\}.$$

Example 1.

Let

$A = (-30, 84]$, $B = [12, 157)$, $C = \{e, f, g\}$,
$D = \{e, m\}$, $E = \{\{e\}, \{e, f, g\}, e, f, g\}$

Then:

$A \cup B = (-30, 157)$, $\quad A \cap B = [12, 84]$,

$A - B = (-30, 12)$, $\quad B - A = (84, 157)$,

$A \times B = \{(x,y) : -30 < x \leq 84,\ 12 \leq y < 157\}$

(a rectangle with part of its border missing).

$C \cup D = \{e, f, g, m\}$, $\quad C \cap D = \{e\}$,

$C - D = \{f, g\}$, $\quad D - C = \{m\}$,

$C \times D = \{(e,e), (e,m), (f,e), (f,m), (g,e), (g,m)\}$,

$C \cap E = \{e, f, g\}$, $\quad C - E = \emptyset$, $\quad E - C = \{\{e\}, \{e,f,g\}\}$,

etc.

Note: we can generalize Def'ns 1, 2, 4 to more than two sets, e.g.

$C \cup D \cup E = \{e, f, g, m, \{e\}, \{e,f,g\}\}$

$\mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{$ordered quadruples (x,y,z,t) : $x, y, z, t \in \mathbb{R}\}$

(also written $\mathbb{R}^4$), etc.

Generalizing Definition 3 is trickier since, for sets A, B, C, $A - (B - C)$ need not equal $(A - B) - C$. (E.g. consider $A = \{1,2,3,4,5,6\}$, $B = \{1,2,3,4\}$, $C = \{3,4,5,6\}$.)

We can also mix operations, e.g.

$(C \cap D) \cup E = E$, $\quad C \cap (D \cup E) = C$,

etc.

Remark:

In general, we have $A \cup B = B \cup A$ and $A \cap B = B \cap A$, but $A - B \neq B - A$ and $A \times B \neq B \times A$.

Definition 5.

Given a set S, we define the power set $\mathcal{P}(S)$ to be the set of all subsets of S.

Example 2.

For C and D as above,

$$\mathcal{P}(D) = \{\emptyset, \{e\}, \{m\}, \{e,m\}\},$$

$$\mathcal{P}(C) = \{\emptyset, \{e\}, \{f\}, \{g\}, \{e,f\}, \{f,g\}, \{e,g\}, \{e,f,g\}\}.$$

Note that $|\mathcal{P}(D)| = 4$, $|\mathcal{P}(C)| = 8$.

In general, we have

Theorem. If S is a finite set (that is, $|S| = n$ for some $n \in \mathbb{N}$), then

$$|\mathcal{P}(S)| = 2^{|S|}.$$

(Proof later.)

For example, for E as above,

$$|\mathcal{P}(E)| = 2^{|E|} = 2^5 = 32.$$

Note: if S is infinite, then $\mathcal{P}(S)$ is huge - even larger than S, in a certain sense.

E.g. $|\mathcal{P}(\mathbb{Z})| = |\mathbb{R}|$, in a sense we'll discuss later.