

- Recall that the addition principle (AP), together with the Inclusion-Exclusion Principle (IEP), say: to count the number of things with Property 1 or Property 2, or both, add the counts of each, but then you've counted the number that have *both* properties twice, so to compensate, subtract that number once.

Let's interpret this in terms of *sets*. Fill in the blanks; each blank should involve only of the symbols A , B , $\cup$, $\cap$, and $| \cdot |$ (that is, the vertical bars used to denote the number of elements in a set), perhaps in combination with each other.

Let A be the set of items with Property 1, and let B be the set of items with Property 2. Then the set of items with one property or the other (or both) is the set $A \cup B$. To count the elements in this set, we count the number of elements of A , add to this the number of elements of B , and then, to compensate for overcounting, we subtract the number of elements in $A \cap B$. In other words, we have the formula

$$|A \cup B| = |A| + \underline{|B|} - \underline{|A \cap B|}.$$

- Let's generalize the above to three sets. Fill in the blanks with the symbols A , B , C , $\cup$, $\cap$, and $| \cdot |$, perhaps in combination with each other.

Let A be the set of items with property 1; let B be the set of items with Property 2; let C be the set of items with Property 3. Then the set of items with at least one of the three properties is the set $A \cup B \cup C$. To count the elements in this set, we count the number of elements of A , add to this the number of elements of B , and add to this the number of elements of C . But we've overcompensated: we've counted elements of $A \cap B$, and of $A \cap C$, and of $B \cap C$, twice. To compensate, we subtract the number of elements in each of these sets.

But wait, we're not done! By adding $|A|$ to $|B|$ to $|C|$, we've counted the elements in $A \cap B \cap C$ three times. Moreover, in subtracting $|A \cap B|$ and $|A \cap C|$ and $|B \cap C|$, we've subtracted off the elements in $A \cap B \cap C$ three times. So the net effect is to count the elements in $A \cap B \cap C$ zero times. But we don't want to count them that many times, we want to count them exactly once. So let's put them back!

In other words, all told, we have the formula

$$\begin{aligned}
 |A \cup B \cup C| &= |A| + \underline{|B|} + \underline{|C|} \\
 &\quad - |A \cap B| - \underline{|A \cap C|} - \underline{|B \cap C|} \\
 &\quad + \underline{|A \cap B \cap C|}.
 \end{aligned}$$

3. Use the formula from the previous problem to solve (the last part of) the following problem: in a standard 52-card deck, how many 5-card lists:

- (a) Have an ace as the first card;

$$4 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = 499,752 \text{ (we did this one for you, to get you started);}$$

- (b) Contain no face cards (that is, Jacks, Queens, or Kings);

$$\underline{40 \cdot 39 \cdot 38 \cdot 37 \cdot 36 = 23,990,400}$$

- (c) Are all of the same suit;

$$\underline{52 \cdot 12 \cdot 11 \cdot 10 \cdot 9 = 617,760}$$

- (d) Have an ace as the first card *and* contain no face cards;

$$\underline{4 \cdot 39 \cdot 38 \cdot 37 \cdot 36 = 7,896,096}$$

- (e) Have an ace as the first card *and* are all of the same suit;

$$\underline{4 \cdot 12 \cdot 11 \cdot 10 \cdot 9 = 47,520}$$

- (f) Contain no face cards *and* are all of the same suit;

$$\underline{40 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 120,960}$$

- (g) Have an ace as the first card *and* contain no face cards *and* are all of the same suit;

$$\underline{4 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 12,096}$$

- (h) Have an ace as the first card *or* contain no face cards *or* are all of the same suit.

$$\begin{aligned} &4 \cdot 51 \cdot 50 \cdot 49 \cdot 48 + 40 \cdot 39 \cdot 38 \cdot 37 \cdot 36 + 52 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \\ &- 4 \cdot 39 \cdot 38 \cdot 37 \cdot 36 - 4 \cdot 12 \cdot 11 \cdot 10 \cdot 9 - 40 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \\ &+ 4 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 85,986,816. \end{aligned}$$