Solutions to Selected Exercises, HW #9

Assignment:

- T-BOP Chapter 10 (pages 196-197): Exercises 25, 27, 28, 30.

Exercise 25. Concerning the Fibonacci sequence, prove that

$$F_1 + F_2 + F_3 + F_4 + \cdots + F_n = F_{n+2} - 1.$$

Proof. Let $A(n)$ be the identity in question.

Step 1: Is $A(1)$ true?

$$\begin{aligned} F_1 &\stackrel{?}{=} F_3 - 1 \\ 1 &= 2 - 1, \end{aligned}$$

so $A(1)$ is true.

Step 2: Assume.

$$A(k) : F_1 + F_2 + F_3 + F_4 + \cdots + F_k = F_{k+2} - 1.$$

Then by the formula $F_{n+2} = F_{n+1} + F_n$ and the induction hypothesis,

$$\begin{aligned} &F_1 + F_2 + F_3 + F_4 + \cdots + F_{k+1} \\ &= (F_1 + F_2 + F_3 + F_4 + \cdots + F_k) + F_{k+1} \\ &= F_{k+2} - 1 + F_{k+1} \\ &= F_{k+3} - 1, \end{aligned}$$

so $A(k+1)$ follows. Therefore, $A(k) \Rightarrow A(k+1)$.

So by the principle of mathematical induction, $A(n)$ is true $\forall n \in \mathbb{N}$. $\square$

Exercise 27. Concerning the Fibonacci sequence, prove that

$$F_1 + F_3 + F_5 + F_7 + \cdots + F_{2n-1} = F_{2n}.$$

Proof. Let $A(n)$ be the identity in question.

Step 1: Is $A(1)$ true?

$$\begin{aligned} F_1 &\stackrel{?}{=} F_{2\cdot 1} \\ 1 &= 1, \end{aligned}$$

so $A(1)$ is true.

Step 2: Assume.

$$A(k) : F_1 + F_2 + F_3 + F_4 + \cdots + F_{2k-1} = F_{2k}.$$

Then by the formula $F_{n+2} = F_{n+1} + F_n$ and the induction hypothesis,

$$\begin{aligned}
&F_1 + F_2 + F_3 + F_4 + \cdots + F_{2(k+1)-1} \\
&= (F_1 + F_2 + F_3 + F_4 + \cdots + F_{2k-1}) + F_{2k+1} \\
&= F_{2k} + F_{2k+1} \\
&= F_{2k+2} = F_{2(k+1)},
\end{aligned}$$

so $A(k+1)$ follows. Therefore, $A(k) \Rightarrow A(k+1)$.

So by the principle of mathematical induction, $A(n)$ is true $\forall n \in \mathbb{N}$.

Exercise 28. Concerning the Fibonacci sequence, prove that

$$F_2 + F_4 + F_6 + F_8 + \cdots + F_{2n} = F_{2n+1} - 1.$$

Proof. Let $A(n)$ be the identity in question.

Step 1: Is $A(1)$ true?

$$\begin{aligned}
F_2 &\stackrel{?}{=} F_3 - 1 \\
1 &= 2 - 1,
\end{aligned}$$

so $A(1)$ is true.

Step 2: Assume.

$$A(k) : F_2 + F_4 + F_6 + F_8 + \cdots + F_{2k} = F_{2k+1} - 1.$$

Then by the formula $F_{n+2} = F_{n+1} + F_n$ and the induction hypothesis,

$$\begin{aligned}
&F_2 + F_4 + F_6 + F_8 + \cdots + F_{2(k+1)} \\
&= (F_2 + F_4 + F_6 + F_8 + \cdots + F_{2k}) + F_{2k+2} \\
&= F_{2k+1} - 1 + F_{2k+2} \\
&= F_{2k+3} - 1 = F_{2(k+1)+1} - 1,
\end{aligned}$$

so $A(k+1)$ follows. Therefore, $A(k) \Rightarrow A(k+1)$.

So by the principle of mathematical induction, $A(n)$ is true $\forall n \in \mathbb{N}$. $\square$

Exercise 30. Here F_n is the nth Fibonacci number. Prove that

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

Proof. Let $A(n)$ be the statement of the proposition.

Step 1: Is $A(1)$ true?

$$F_1 \stackrel{?}{=} \frac{\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1}{\sqrt{5}}$$
$$1 \stackrel{?}{=} \frac{\frac{1+\sqrt{5}-1+\sqrt{5}}{2}}{\sqrt{5}}$$
$$1 = \frac{\sqrt{5}}{\sqrt{5}}.$$

So $A(1)$ is true.

Step 2: Is $A(2)$ true?

$$F_2 \stackrel{?}{=} \frac{\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2}{\sqrt{5}}$$

Using the hint in S-POP, which says $\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2}$ and $\left(\frac{1-\sqrt{5}}{2}\right)^2 = \frac{3-\sqrt{5}}{2}$, we find that the question is

$$F_2 \stackrel{?}{=} \frac{\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{3-\sqrt{5}}{2}\right)}{\sqrt{5}}$$
$$1 \stackrel{?}{=} \frac{\frac{3+\sqrt{5}-3+\sqrt{5}}{2}}{\sqrt{5}}$$
$$1 = \frac{\sqrt{5}}{\sqrt{5}}.$$

So $A(2)$ is true.

Step 3: Assume

$$A(k) : F_k = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k}{\sqrt{5}}$$

and

$$A(k+1) : F_{k+1} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}}.$$

Then

$$\begin{aligned}
F_{k+2} &= F_{k+1} + F_k \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k}{\sqrt{5}} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1+\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}} - \frac{\left(\frac{1-\sqrt{5}}{2}\right)^k + \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}}{\sqrt{5}} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k \left(1 + \frac{1+\sqrt{5}}{2}\right)}{\sqrt{5}} - \frac{\left(\frac{1-\sqrt{5}}{2}\right)^k \left(1 + \frac{1-\sqrt{5}}{2}\right)}{\sqrt{5}} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k \left(\frac{3+\sqrt{5}}{2}\right)}{\sqrt{5}} - \frac{\left(\frac{1-\sqrt{5}}{2}\right)^k \left(\frac{3-\sqrt{5}}{2}\right)}{\sqrt{5}} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^k \left(\frac{1+\sqrt{5}}{2}\right)^2}{\sqrt{5}} - \frac{\left(\frac{1-\sqrt{5}}{2}\right)^k \left(\frac{1-\sqrt{5}}{2}\right)^2}{\sqrt{5}} \\
&= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+2}}{\sqrt{5}} - \frac{\left(\frac{1-\sqrt{5}}{2}\right)^{k+2}}{\sqrt{5}},
\end{aligned}$$

so $A(k+2)$ follows. So by the principle of double whammy mathematical induction, $A(n)$ is true $\forall n \in \mathbb{N}$. $\square$