Solutions to Selected Exercises, HW #8

Assignment:

- S-POP Part B(v) : Exercises B(v) 1, 3, 6.
- T-BOP Chapter 10 (page 195): Exercises 3, 4, 8, 12, 13.

S-POP, Part B(v)

Exercise 3. Use mathematical induction to prove that, for any positive integer n,

$$\frac{d}{dx}x^n = nx^{n-1}$$

(pretend you didn't already know this, although it's OK to assume it's true for $n = 1$). Hint: for the inductive step, use the product rule.

Proof. Let $A(n)$ be the statement

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Step 1: Is $A(1)$ true?

$$\frac{d}{dx}x^1 = \frac{d}{dx}x = 1 = 1 \cdot x^{1-1},$$

so $A(1)$ is true.

Step 2: Assume

$$A(k) : \frac{d}{dx}x^k = kx^{k-1}.$$

Then by the product rule and the induction hypothesis,

$$\begin{aligned} \frac{d}{dx}x^{k+1} &= \frac{d}{dx}x^k \cdot x \\ &= \left(\frac{d}{dx}x^k\right)\cdot x + \left(\frac{d}{dx}x\right)\cdot x^k \\ &= \left(kx^{k-1}\right)\cdot x + \left(1\right)\cdot x^k \\ &= kx^k x + x^k = (k+1)x^k, \end{aligned}$$

so $A(k+1)$ follows. Therefore, $A(k) \Rightarrow A(k+1)$.

So by the principle of mathematical induction, $A(n)$ is true $\forall n \in \mathbb{N}$. $\square$

Exercise 6. Let A_n be the statement

$$1 + 2 + 3 + \cdots + n = \frac{(2n+1)^2}{8}.$$

Prove that if $A(k)$ is true for any positive integer k, then so is $A(k+1)$. Is $A(n)$ true for all positive integers n? Explain your answer.

Solution. First we prove that $A(k) \Rightarrow A(k+1)$. So assume

$$A(k) : 1 + 2 + 3 + \cdots + k = \frac{(2k+1)^2}{8}.$$

Then

$$\begin{aligned} 1 + 2 + 3 + \cdots + (k+1) &= (1 + 2 + 3 + \cdots + k) + k + 1 \\ &= \frac{(2k+1)^2}{8} + k + 1 = \frac{(2k+1)^2}{8} + \frac{8(k+1)}{8} \\ &= \frac{(2k+1)^2 + 8(k+1)}{8} = \frac{4k^2 + 4k + 1 + 8k + 8}{8} = \frac{4k^2 + 12k + 9}{8} \\ &= \frac{(2k+3)^2}{8} = \frac{(2(k+1)+1)^2}{8}, \end{aligned}$$

so $A(k+1)$ follows. Therefore, $A(k) \Rightarrow A(k+1)$.

However, $A(n)$ is *not* true for all $n \in \mathbb{N}$. For example, $A(1)$ is the statement 1=9/8, which is false.

The point of this exercise is that it's not enough just to show that $A(k) \Rightarrow A(k+1)$. You also need to prove the base case $A(1)$. Without that, the statement $A(n)$ might not even be true for a single positive integer n.

T-BOP Chapter 10 (page 195)

Prove the following statements with either induction, strong induction, or proof by smallest counterexample.

Exercise 4. If $n \in N$, then

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

Proof. Let $A(n)$ be the statement of the proposition.

Step 1: Is $A(1)$ true?

$$\begin{aligned} 1 \cdot 2 &\stackrel{?}{=} \frac{1(1+1)(1+2)}{3} \\ 2 &= 2. \end{aligned}$$

so $A(1)$ is true.

Step 2: Assume

$$A(k) : 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}.$$

Then

$$\begin{aligned}
&1\cdot 2+2\cdot 3+3\cdot 4+4\cdot 5+\cdots+(k+1)(k+1+1)\\
&=\big(1\cdot 2+2\cdot 3+3\cdot 4+4\cdot 5+\cdots+k(k+1)\big)+(k+1)(k+2)\\
&=\frac{k(k+1)(k+2)}{3}+(k+1)(k+2)=\frac{k(k+1)(k+2)+3(k+1)(k+2)}{3}\\
&=\frac{(k+1)(k+2)(k+3)}{3}=\frac{(k+1)((k+1)+1)((k+1)+2)}{3},
\end{aligned}$$

so $A(k+1)$ follows. Therefore, $A(k)\Rightarrow A(k+1)$.

So by the principle of mathematical induction, $A(n)$ is true $\forall n\in\mathbb{N}$. $\square$

Exercise 8. If $n\in N$, then

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{n}{(n+1)!}=1-\frac{1}{(n+1)!}.$$

Proof. Let $A(n)$ be the statement of the proposition.

Step 1: Is $A(1)$ true?

$$\begin{aligned}
\frac{1}{2!}&\stackrel{?}{=}\frac{1}{(1+1)!}\\
\frac{1}{2}&=\frac{1}{2}.
\end{aligned}$$

so $A(1)$ is true.

Step 2: Assume

$$A(k):\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{k}{(k+1)!}=1-\frac{1}{(k+1)!}.$$

Then

$$\begin{aligned}
&\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{k+1}{((k+1)+1)!}\\
&=\left(\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{k}{(k+1)!}\right)+\frac{k+1}{((k+1)+1)!}\\
&=1-\frac{1}{(k+1)!}+\frac{k+1}{(k+2)!}=1-\frac{k+2}{(k+2)!}+\frac{k+1}{(k+2)!}\\
&=1+\frac{-(k+2)+k+1}{(k+2)!}=1-\frac{1}{(k+2)!},
\end{aligned}$$

so $A(k+1)$ follows. (To get a common denominator, we have used the fact $1/(k+1)!=(k+2)/(k+2)!$.) Therefore, $A(k)\Rightarrow A(k+1)$.

So by the principle of mathematical induction, $A(n)$ is true $\forall n\in\mathbb{N}$. $\square$

Exercise 12. Prove that $9|(4^{3n}+8)$ for every integer $n \geq 0$.

Proof. Let $A(n)$ be the statement $9|(4^{3n}+8)$.

Step 1: Is $A(0)$ true?

$$\begin{aligned}&9|(4^{3\cdot 0}+8)\ ?\\&9|(4^0+8)\ ?\\&9|9.\end{aligned}$$

So $A(0)$ is true.

Step 2: Assume $A(k) : 9|(4^{3k}+8)$. Then

$$\begin{aligned}4^{3(k+1)}+8 &= 4^{3k+3}+8+4^{3k}-4^{3k}\\&= 4^{3k}+8+4^{3k+3}-4^{3k}\\&= 4^{3k}+8+4^{3k}\left(4^3-1\right)\\&= 4^{3k}+8+4^{3k}\cdot 63.\end{aligned}$$

Now $9|(4^{3k}+8)$ by the induction hypothesis, and $9|\left(4^{3k}\cdot 63\right)$ since $63 = 9\cdot 7$. So $9|\left(4^{3k}+8+4^{3k}\cdot 63\right)$ (we've used Exercise B(i)-(3) in S-POP). So $A(k+1)$ follows.

So by the principle of mathematical induction, $A(n)$ is true $\forall$ integers $n \geq 0$. $\square$

Exercise 13. Prove that $6|(n^3-n)$ for any $n \in \mathbb{N}$.

Proof. Let $A(n)$ be the statement $6|(n^3-n)$.

Step 1: Is $A(1)$ true?

$$\begin{aligned}&6|(1^3-1)\ ?\\&6|0.\end{aligned}$$

So $A(1)$ is true.

Step 2: Assume $A(k) : 6|(k^3-k)$. Then

$$\begin{aligned}(k+1)^3-(k+1) &= k^3+3k^2+3k+1-k-1\\&= k^3+3k^2+3k-k\\&= (k^3-k)+3k^2+3k\\&+ (k^3-k)+3(k^2+k).\end{aligned}\tag{1}$$

Now note that $k^2+k = k(k+1)$. Since either k or $k+1$ must be even (given any two consecutive integers, either the first one or the second one is even), we can write $k^2+k = 2m$ where $m \in \mathbb{Z}$. So equation (1) gives

$$(k+1)^3-(k+1) = (k^3-k)+3\cdot 2m = (k^3-k)+6m.$$

Now $6|(k^3-k)$ by the induction hypothesis, and clearly $6|6m$. So $6|\left((k^3-k)+6m\right)$, and $A(k+1)$ follows. (Again, we've used Exercise B(i)-(3) in S-POP.)

So by the principle of mathematical induction, $A(n)$ is true $\forall n \in \mathbb{N}$. $\square$