Solutions to Selected Exercises, HW #7

Assignment (all in T-BOP):

- Section 3.5, page 89: Exercises 1, 2, 3, 5, 6, 8, 10, 17.
- Section 3.6, page 92: Exercises 8, 10, 12.
- Section 3.7, page 95: Exercises 1, 7, 8, 14.

T-BOP, Section 3.5

Exercise 2. Suppose A is a set for which $|A| = 100$. How many subsets of A have 5 elements? How many subsets have 10 elements? How many have 99 elements?

Solution. The number of subsets with 5, 10, and 99 elements respectively is

$$\binom{100}{5} = \frac{100!}{5! \cdot 95!} = \frac{100 \cdot 99 \cdot 98 \cdot 97 \cdot 96}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 75{,}287{,}520,$$
$$\binom{100}{10} = \frac{100!}{10! \cdot 90!} = \frac{100 \cdot 99 \cdot 98 \cdot 97 \cdot 96 \cdot 95 \cdot 94 \cdot 93 \cdot 92 \cdot 91}{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 17{,}310{,}309{,}456{,}440,$$
$$\binom{100}{99} = \frac{100!}{99! \cdot 1!} = \frac{100}{1} = 100.$$

Exercise 6. $\big|X \in \mathscr{P}(\{0,1,2,3,4,5,6,7,8,9\}) : |X| = 4\}\big| =$

Solution. The notation is really daunting, but all this problem is asking is: How many subsets X of $\{0,1,2,3,4,5,6,7,8,9\}$ have 4 elements? Since $\{0,1,2,3,4,5,6,7,8,9\}$ has 10 elements, the answer is

$$\binom{10}{4} = 210.$$

Exercise 8. This problem concerns lists made from the symbols $A, B, C, D, E, F, G, H, I$.

(a) How many length-5 lists can be made if there is no repetition and the list is in alphabetical order? (Example: BDEFI or ABCGH, but not BACGH.)

Solution. This part is actually trickier than part (b). The hint I gave to this problem says:

> Part (a) of this problem seems like a list problem instead of a set problem. But the requirement of alphabetical order means that, out of all possible lists that contain a given set of letters, you can only choose one (namely, the one that's in alphabetical order). So you have to divide your list count by something, just as we did when counting sets.

The idea is this: we know that there are $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$ length-5 lists from the 9 letters A through I. But, out of all possible lists that contain a given set of letters, you can only choose one (namely, the one that's in alphabetical order). So, to account for this overcounting, we need to divide by the number of length-5 lists that give the same alphabetically ordered list. This number is 5!. So the answer to this problem is

$$\frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{5!} = 126.$$

Note that this is the same as $\binom{9}{5}$.

(b) How many length-5 lists can be made if repetition is not allowed and the list is not in alphabetical order?

Solution. This part is easy: it's just

$$9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 15{,}120.$$

Exercise 10. A department consists of 5 men and 7 women. From this department you select a committee with 3 men and 2 women. In how many ways can you do this?

Solution.

$$\binom{5}{3} \cdot \binom{7}{2} = 210.$$

T-BOP, Section 3.6

Exercise 8. Use Fact 3.5 (page 87) to derive Equation (3.3) (page 90).

Solution. We have

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}.$$

Multiply the first rracfion in the right by k/k and the second fraction on the right by $(n-k+1)/(n-k+1)$. As described in the hints, this gives

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n! \cdot k}{k!(n-k+1)!} + \frac{n! \cdot (n-k+1)}{k!(n-k+1)!} \\
&= \frac{n! \cdot (k+n-k+1)}{k!(n-k+1)!} \\
&= \frac{n! \cdot (n+1)}{k!(n-k+1)!} \\
&= \frac{(n+1)!}{k!(n-k+1)!} = \binom{n+1}{k}.
\end{aligned}$$

Exercise 10. Show that the formula $k\binom{n}{k} = n\binom{n-1}{k-1}$ is true for all integers n,k with $0 \leq k \leq n$.

Solution.

$$k\binom{n}{k} = k \cdot \frac{n!}{k!(n-k)!} = \frac{n!}{(k-1)!(n-k)!},$$

since $k/k! = 1/(k-1)!$. On the other hand,

$$n\binom{n-1}{k-1} = n \cdot \frac{(n-1)!}{(k-1)!(n-1-(k-1))!} = \frac{n!}{(k-1)!(n-k)!},$$

since $n \cdot (n-1)! = n!$ and $n-1-(k-1) = n-k$. So the two quantities are equal.

Exercise 12. Show that $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$.

Solution. We have

$$\binom{n}{k}\binom{k}{m} = \frac{n!}{k!(n-k)!} \cdot \frac{k!}{m!(k-m)!} = \frac{n!}{(n-k)!m!(k-m)!}.$$

Multiply top and bottom by $(n-m)!$ to get

$$\begin{aligned}\binom{n}{k}\binom{k}{m} &= \frac{n!(n-m)!}{(n-m)!(n-k)!m!(k-m)!} \\ &= \frac{n!}{m!(n-m)!} \cdot \frac{(n-m)!}{(k-m)!(n-m-(k-m))!} = \binom{n}{m}\binom{n-m}{k-m}.\end{aligned}$$

T-BOP, Section 3.7

Exercise 8. Consider 4-card hands dealt off of a standard 52-card deck. How many hands are there for which all 4 cards are of different suits or all 4 cards are red?

Solution. First, we count the number of hands for which all 4 cards are of different suits. To do these, we can choose one heart, then one diamond, then one spade, then one club, yielding

$$13 \cdot 13 \cdot 13 \cdot 13 = 13^4 = 28{,}561$$

possible choices.

Next, we count the number of hands for which all four cards are red. There are $13 + 13 = 26$ red cards, so the number of hands for which all four cards are red is

$$\binom{26}{4} = 14{,}950.$$

Next, we count the number of hands that are all of different suits, *and* for which all four cards are red. But this is not possible: if they are all of different suits, then one is a spade and one is a club, so they can't all be red.

So, by the inclusion-exclusion principle, the number of hands that are all of a different suit, *or* for which all four cards are red, is

$$28{,}561 + 14{,}950 - 0 = 43{,}511.$$

Exercise 14. How many 3-card hands (from a standard 52-card deck) have the property that it is not the case that all cards are black or all cards are of the same suit?

Solution. The number of 3-card hands that are all black is

$$\binom{26}{3} = 2{,}600.$$

The number that are all of the same suit is

$$4 \cdot \binom{13}{3} = 1{,}144.$$

The number that are all black *and* all of the same suit is

$$2 \cdot \binom{13}{3} = 572.$$

So the number that are all black *or* all of the same suit is

$$2{,}600 + 1{,}144 - 527 = 3{,}217.$$

So the number that DON'T have the latter property is

$$\binom{52}{3} - 2{,}600 + 1{,}144 - 527 = 16{,}068.$$