Take-Home Midterm Exam, due by the start of class on Friday, April 5

You are to complete this midterm on your own, without assistance from any written, human, AI, web, or other resources, **except that you may refer to:** class notes (which are posted on our Canvas page), any solutions or fact sheets posted on our Canvas page, T-BOP, S-POP, yourself, and me. (You are free to ask me questions, most of which I might not answer completely for you.) Of course, you can also use paper and something to write with. Please do not use a calculator (you won't need one).

Complete this exam on your own paper. Be *neat*. **You must complete, and attach to your exam, a copy the cover sheet at the end of this exam.**

Your exam may be submitted on hard copy paper or online, but either way, it **must** be turned by at the *beginning* of class on Friday, April 5. Late exams will not be accepted.

Good luck!

(Exam begins on next page)

1 (7 points) T-BOP Section 1.8 (p. 29): Exercise 14. In particular: if the given statement

$$\bigcap_{\alpha \in I} A_\alpha \subseteq \bigcap_{\alpha \in J} A_\alpha$$

is true, prove it. If it's false, provide a counterexample (and explain why it's a counterexample).

2 (15 points) Quantifiers and negations. For this exercise, you might want to recall that the negation of a statement like "$\exists x \in X \colon Q(x)$" is the statement "$\forall x \in X$, $\sim Q(x)$." Here, $\sim Q(x)$ denotes the negation of $Q(x)$; that is, $\sim Q(x)$ means "not $Q(x)$." (In other words, $\sim Q(x)$ means "$Q(x)$ is false.") Similarly, the negation of a statement like "$\forall x \in X \colon Q(x)$" is the statement "$\exists x \in X$, $\sim Q(x)$."

(a) Express the negation of the statement $\forall x \in X, \exists y \in Y : Q(x, y)$ in terms of $\sim Q(x, y)$. (Here, $Q(x, y)$ is some statement involving objects x, y.)

(b) Express the negation of the statement $\forall x \in X, \exists y \in Y, \forall z \in Z : Q(x, y, z)$ in terms of $\sim Q(x, y, z)$. (Here, $Q(x, y, z)$ is some statement involving objects x, y, z.)

(c) One way of defining limits is as follows: we say

$$\lim_{n \to \infty} x_n = L$$

if

$$\forall \varepsilon > 0,\ \exists R \in \mathbb{R}, \forall n > R, |x_n - L| < \varepsilon. \qquad (*)$$

(This looks just a bit different from the definition in class, and in S-POP, but it amounts to the same thing.) Now, form the negation of $(*)$. In other words, express the statement

$$\lim_{n \to \infty} x_n \neq L$$

in terms of quantifiers, ε, n, $\mathbb{R}$, and $|x_n - L|$. Hint 1: part (b) of this exercise may be of help. Hint 2: your final answer should involve the inequality $|x_n - L| \geq \varepsilon$.

3 (15 points) More quantifiers. Identify each of the following statements as true or false (circle "**T**" or "**F**"). **Please explain your answers:** If a statement is true, explain why (you don't need to provide a complete proof; just a sentence or two will do). If a statement is false, provide a counterexample to the statement, and explain why it's a couterexample.

(a) $\forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}\colon (m-n)|k.$ **T** **F**

(b) $\exists k \in \mathbb{Z}\colon \forall m \in \mathbb{Z}, \forall n \in \mathbb{Z}, (m-n)|k.$ **T** **F**

(c) $\sim\big(\forall m \in \mathbb{Z}, \exists k \in \mathbb{Z}\colon \forall n \in \mathbb{Z}, (m-n)|k\big).$ **T** **F**

4 (20 points) Set proof. For this exercise you may use the definition, from pp. 7 of S-POP, of the symmetric difference $C\Delta D$ of two sets C and D:

$$C\Delta D = (C - D) \cup (D - C).$$

Supply a complete proof of the following proposition, by showing that the set defined on the left hand side of the equal sign is contained in the set on the right, and vice versa. **NOTE:** You may use any results proved in S-POP, T-BOP, our lecture notes, or any homework assignment or exam, provided you cite those results. (For example, "We know, by Proposition B(ii)-1_E of S-POP, that $S \cap T \subseteq S$ for any sets S and T." I'm not saying that result will be relevant here; I'm just providing an example of how to cite a result.)

Proposition. For any sets B, C, and D, we have

$$B \cap (C\Delta D) = (B \cap C)\Delta(B \cap D).$$

5 (12 points) Prove *carefully* that

$$\lim_{n\to\infty} \frac{4n+3}{n+3} = 4.$$

Use only the definition of limit from S-POP: we say

$$\lim_{n\to\infty} x_n = L$$

if

$$\forall \varepsilon > 0, \exists R \in \mathbb{R}, n > R \Rightarrow |x_n - L| < \varepsilon.$$

Feel free to include your "scratchwork" in your answer; if you do, please put your scratchwork in square brackets ([ ]).

6 (16 points) Let a be a real number with $a \neq 1$. Use the Principle of Mathematical Induction to show that, for all $n \in \mathbb{N}$,

$$a^1 + a^2 + a^3 + \cdots + a^n = \frac{a^{n+1} - a}{a - 1}.$$

7 (15 points) (a) Let $n \in \mathbb{Z}$. Prove carefully that $3|n$ and $5|n \Leftrightarrow 15|n$. Hint for showing the "$\Rightarrow$" direction: since $5|n$, we can write $n = 5m$ for some $m \in \mathbb{Z}$. Noting that $5 = 3 \cdot 2 - 1$, we have $n = (3 \cdot 2 - 1) \cdot m = 3(2m) - m$. Now use parts of S-POP Exercise B(i)-3 (cite them carefully), together with the assumption that $3|n$, to conclude that $3|m$. Then explain why this gives you your desired result.

(b) Let $n \in \mathbb{Z}$. If $6|n$ and $4|n$, does it follow that $24|n$? If yes, prove it. If no, give a counterexample, and explain why it's a counterexample.

(Don't forget the cover page below.)

MATH 2001-004: Intro to Discrete Math

April 5, 2024

TAKE-HOME MIDTERM EXAM

I have neither given nor received unauthorized assistance on this exam.

Name:____

Signature:____

DO NOT WRITE IN THIS BOX!

Problem	Points	Score
1	7 pts	
2	15 pts	
3	15 pts	
4	20 pts	
5	12 pts	
6	16 pts	
7	15 pts	
TOTAL	100 pts	