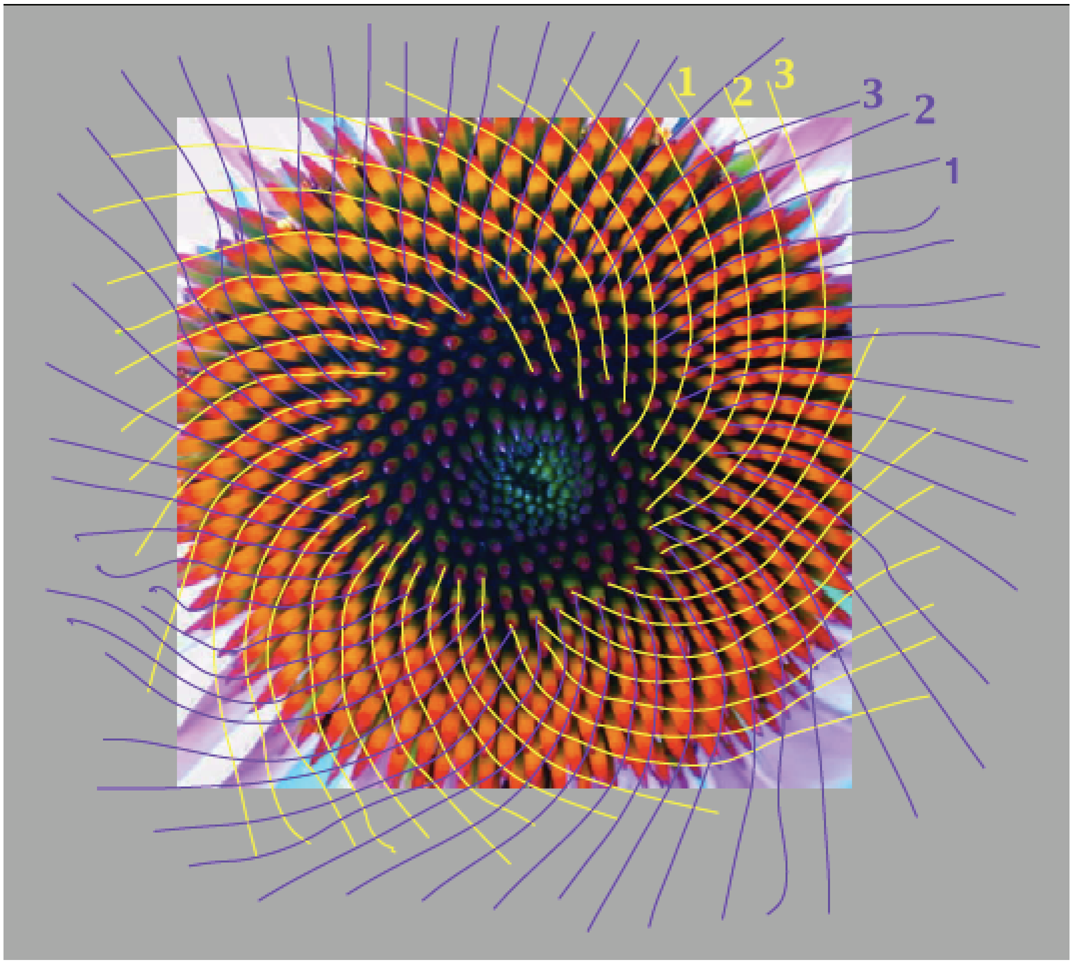
Fibonacci numbers and the golden ratio (SOLUTIONS)

Exercise 1. Carefully count the number of clockwise (yellow) and counterclockwise (purple) spiral arms in the coneflower below.



Clockwise spiral arms: 34 Counterclockwise spiral arms: 55

Now recall the *Fibonacci sequence* F_n, which looks like this:

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots.$$

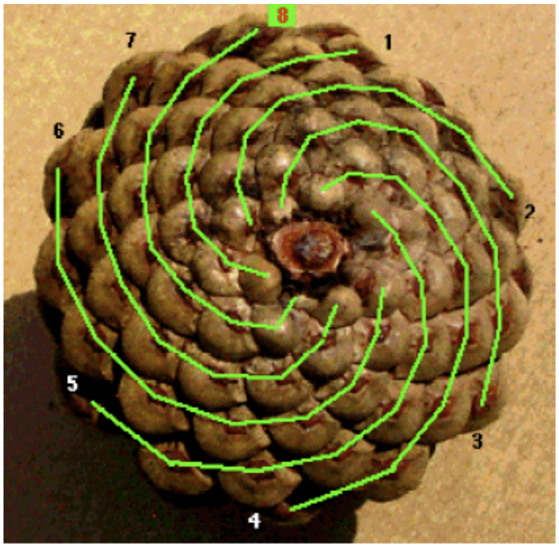
As we saw in class, this sequence is defined by

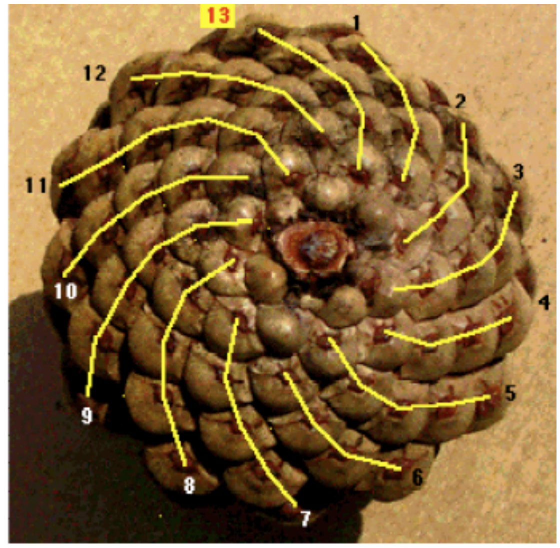
$$F_1 = F_2 = 1, \qquad (*)$$

$$F_{n+2} = F_{n+1} + F_n \quad (n \geq 1). \qquad (*')$$

You may have noticed that the numbers of spiral arms of the above coneflower are Fibonacci numbers!

FACT: Fibonacci numbers are EVERYWHERE. For example, count clockwise and counter-clockwise spiral arms on a pine cone:





You'll get consecutive Fibonacci numbers! Similar things happen with sunflowers, pineapples, broccoli florets, etc. See

`https://en.wikipedia.org/wiki/Fibonacci_number`

Exercise 2. Write down the next nine Fibonacci numbers after those listed on the previous page.

34, 55, 89, 144, 233, 377, 610, 987, 1597,...

Particularly interesting things happen when we examine *ratios* of successive Fibonacci numbers. Let's do this.

We define a sequence R_n by

$$R_n = \frac{F_{n+1}}{F_n}$$

for $n \geq 1$ (F_n denotes the nth Fibonacci number, as above). So the sequence R_n starts like this:

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \ldots.$$

Exercise 3. Write down the next nine R_n's as fractions. Then write these nine terms as decimal numbers, with at least four places after the decimal point. Do the R_n's appear to be converging? That is, do they appear to have a limit? If so, what (approximately) does this limit appear to be (to as many decimal places as you care to speculate)?

$$\frac{21}{13} = 1.615385, \quad \frac{34}{21} = 1.619048, \quad \frac{55}{34} = 1.617647, \quad \frac{89}{55} = 1.618182, \quad \frac{144}{89} = 1.617976,$$

$$\frac{233}{144} = 1.618056, \quad \frac{377}{233} = 1.618026, \quad \frac{610}{377} = 1.608037, \quad \frac{987}{610} = 1.618033, \ldots.$$

The R_n's seem to be bouncing up and down around some number that seems to be around 1.61803.

The number to which your above R_n's converge is, actually, a number that shows up in various other places too.

In the next problem, we investigate several of those places, using your provided rulers.

Exercise 4(a) Take out a BuffOne Card or credit card or driver's license (just one card per group is fine). Measure, in millimeters, the length of the longer side, and the length of shorter side, of you card. Record these measurements, as well as the *proportion*, meaning longer length divided by shorter length, as a decimal to a few places, of your card here:

Longer: 86 Shorter: 54 Proportion: 1.59259

Now repeat the above measurements for several objects from around the room. Specifically, measure:

(b) The rectangular plate surrounding the up/down projector screen controls, located on the left-hand wall near the front of the room:

Longer: 129 Shorter: 84 Proportion: 1.53571

(c) The rectangular screen of the projector controller, located on Dr. S's desk at the front of the room:

Longer: 175 Shorter: 117 Proportion: 1.49573

(d) A single-switch switchplate, like this: . I don't think there's one in this room, but you can look for one in this building somewhere. Just please don't enter any in-use classrooms or otherwise disturb anyone, and come back quickly.

Longer: 115 Shorter: 71 Proportion: 1.61972

Exercise 5. The number $(1+\sqrt{5})/2$, often called the *golden mean* or the *golden ratio*, and often denoted by Φ, is special. It shows up in many real-life, and mathematical, situations. What are some such situations? To answer, plug this number into your calculator, and evaluate as a decimal to a few decimal places. How does what you get compare to some of the numbers above? See especially Exercises 3 and 4 above.

$$\frac{1+\sqrt{5}}{2} \approx 1.618033989.$$

This number is about what the ratios R_n of successive Fibonacci numbers seem to be converging to. It's also not far off from the physical ratios we measured above (especially the switchplate).

Remark. The golden ratio Φ, or numbers close to it, also show up when you divide your height by the height of your belly button; or when you divide the height of your face (chin to crown) by the width of your face; etc. (If you want to take some measurements to test this out, please do so later, outside of class, for the sake of privacy.) There's an awful lot of debate as to whether these phenomena are deeply significant or not. (Perhaps the debate itself makes them significant.)

Exercise 6. Comment on whether this statement seems to be correct: The limit of ratios of consecutive Fibonacci numbers equals the golden ratio. Please explain/justify your answer. No proof is required here; just an intuitive sense based on some of your observations and calculations above. (We'll address this statement rigorously on Monday.)

This seems to be true, since we computed a bunch of R_n's, and they did seem to be approaching 1.61803 or so, whihc is roughly the golden ratio.