Activity: Counting, two ways (and one not-a-way)

In this activity, we're going to count the number of 5-card *lists* (where order matters), from a standard, 52-card deck, that contain *at least one* 3, in two different ways.

You'll see many places where there are blanks like this:

_____ · _____ · _____ · _____ · _____ = _______________.

The idea is: each of the first five, short, blanks should be filled in with a number, between 1 and 52, that indicates how many ways there are of choosing the card in that position on the list. The final, longer blank should be filled in with the product of the five numbers in the shorter blanks.

EXAMPLE: The number of 5-card lists in which the first card is an ace (and the other cards can be anything) is

4 · 51 · 50 · 49 · 48 = 23,990,400.

Now here are your exercises.

Method 1. To count 5-card lists with at least one 3, we'll count those with no 3's, and then use the subtraction principle. (We did this in class Wednesday, but it never hurts to review.)

(a) The *total* number of 5-card lists (of any kind) is

52 · 51 · 50 · 49 · 48 = 311,875,200.

(b) The number of 5-card lists with *no* 3's is

48 · 47 · 46 · 45 · 44 = 205,476,480.

(c) Using your answers to parts (a) and (b) above, and the subtraction principle, we find that the number of 5-card lists with at least one 3 is

311,875,200 − 205,476,480 = 106,398,720.

(over)

Method 2. This time, to count 5-card lists with at least one 3, we'll use the *addition principle.*

(a) The number of 5-card lists where the *first card* is a 3, and the last four cards can be anything, is

$\underline{\ 4\ } \cdot \underline{\ 51\ } \cdot \underline{\ 50\ } \cdot \underline{\ 49\ } \cdot \underline{\ 48\ } = \underline{\ 23{,}990{,}400\ }$.

(b) The number of 5-card lists where the first card is **not** a 3, the second card **is** a 3, and the last three cards can be anything, is

$\underline{\ 48\ } \cdot \underline{\ 4\ } \cdot \underline{\ 50\ } \cdot \underline{\ 49\ } \cdot \underline{\ 48\ } = \underline{\ 22{,}579{,}200\ }$.

(c) The number of 5-card lists where **neither** the first card *nor* the second card is a 3, the third card **is** a 3, and the last two cards can be anything, is

$\underline{\ 48\ } \cdot \underline{\ 47\ } \cdot \underline{\ 4\ } \cdot \underline{\ 49\ } \cdot \underline{\ 48\ } = \underline{\ 21{,}224{,}448\ }$.

(d) The number of 5-card lists where **none of the first three cards** is a 3, the fourth card **is** a 3, and the last card can be anything, is

$\underline{\ 48\ } \cdot \underline{\ 47\ } \cdot \underline{\ 46\ } \cdot \underline{\ 4\ } \cdot \underline{\ 48\ } = \underline{\ 19{,}924{,}992\ }$.

(e) The number of 5-card lists where **none of the first four cards** is a 3, but the fifth card **is** a 3, is

$\underline{\ 48\ } \cdot \underline{\ 47\ } \cdot \underline{\ 46\ } \cdot \underline{\ 45\ } \cdot \underline{\ 4\ } = \underline{\ 18{,}679{,}680\ }$.

(f) Using your answers to parts (a) through (e) above, and the *addition* principle, we find that the number of 5-card lists with at least one 3 is

$\underline{\ 311{,}875{,}200\ } + \underline{\ 205{,}476{,}480\ } + \underline{\ 205{,}476{,}480\ }$

$+ \underline{\ 205{,}476{,}480\ } + \underline{\ 205{,}476{,}480\ } = \underline{\ 106{,}398{,}720\ }$.

NOTE: your answer to part (f) of this exercise should be the same as your answer to part (c) of the exercise on the previous page. If not, please check your work.

(over)

Method 3. The following method *fails*. But let's work it through, and then try to identify the problem.

(a) The number of 5-card lists where the *first card* is a 3 (and where the other four cards can be anything) is

$\underline{\ 4\ } \cdot \underline{\ 51\ } \cdot \underline{\ 50\ } \cdot \underline{\ 49\ } \cdot \underline{\ 48\ } = \underline{\ 23{,}990{,}400\ }$.

(b) The number of 5-card lists where the *second card* is a 3, and the other four cards can be anything, is

$\underline{\ 51\ } \cdot \underline{\ 4\ } \cdot \underline{\ 50\ } \cdot \underline{\ 49\ } \cdot \underline{\ 48\ } = \underline{\ 23{,}990{,}400\ }$.

(c) The number of 5-card lists where the *third card* is a 3, and the other four cards can be anything, is

$\underline{\ 51\ } \cdot \underline{\ 50\ } \cdot \underline{\ 4\ } \cdot \underline{\ 49\ } \cdot \underline{\ 48\ } = \underline{\ 23{,}990{,}400\ }$.

(d) The number of 5-card lists where the *fourth card* is a 3, and the other four cards can be anything, is

$\underline{\ 51\ } \cdot \underline{\ 50\ } \cdot \underline{\ 49\ } \cdot \underline{\ 4\ } \cdot \underline{\ 48\ } = \underline{\ 23{,}990{,}400\ }$.

(e) The number of 5-card lists where the *fifth card* is a 3, and the other four cards can be anything, is

$\underline{\ 51\ } \cdot \underline{\ 50\ } \cdot \underline{\ 49\ } \cdot \underline{\ 48\ } \cdot \underline{\ 4\ } = \underline{\ 23{,}990{,}400\ }$.

(f) If we add the answers to parts (a) through (e) above, to get the number of 5-card lists with at least one 3, we get

$5 \cdot (\underline{\ 23{,}990{,}400\ }) = \underline{\ 119{,}952{,}000\ }$.

(over)

Wrap-up. Method 3 above fails because of overcounting.

(a) Give an example of a 5-card list that's counted exactly **twice** in Method 3. (Please specify both the suit and the face value of each card in this list.)

$3\heartsuit, 3\diamondsuit, 10\spadesuit, \text{A}\clubsuit, 7\heartsuit$

(b) Give an example of a 5-card list that's counted exactly **four times** in Method 3. (Please specify both the suit and the face value of each card in this list.)

$3\heartsuit, 3\diamondsuit, 3\spadesuit, \text{A}\clubsuit, 3\clubsuit$