FINAL EXAM: PRACTICE PROBLEMS

1. Prove that (a) The sum of two odd numbers is even. (b) The product of two odd numbers is odd.

SOLUTION.

(a): We may state this in $P \Rightarrow Q$ form as follows: if $a, b \in \mathbb{Z}$ are odd, then $a + b$ is even.

So assume $a, b \in \mathbb{Z}$ are odd. We may write $a = 2k + 1$ and $b = 2\ell + 1$, where $k, \ell \in \mathbb{Z}$. But then

$$a + b = 2k + 1 + 2\ell + 1 = 2k + 2\ell + 2 = 2(k + \ell + 1) = 2m,$$

where $m = k + \ell + 1 \in \mathbb{Z}$. So, by definition of even integer, $a + b$ is even.

So the sum of two odd numbers is even. $\square$

(b): We may state this in $P \Rightarrow Q$ form as follows: if $a, b \in \mathbb{Z}$ are odd, then ab is odd.

So assume $a, b \in \mathbb{Z}$ are odd. We may write $a = 2k + 1$ and $b = 2\ell + 1$, where $k, \ell \in \mathbb{Z}$. But then

$$ab = (2k + 1)(2\ell + 1) = 4k\ell + 2k + 2\ell + 1 = 2(2k\ell + k + \ell) + 1 = 2m + 1,$$

where $m = 2k\ell + k + \ell \in \mathbb{Z}$. So, by definition of odd integer, ab is odd.

So the product of two odd numbers is odd. $\square$

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2. Prove that, if $m \in 1 + 4\mathbb{Z}$ and $n \in 2 + 4\mathbb{Z}$, then (a) $m + n \in 3 + 4\mathbb{Z}$. (b) $mn \in 2 + 4\mathbb{Z}$.

SOLUTION.

Assume $m \in 1 + 4\mathbb{Z}$ and $n \in 2 + 4\mathbb{Z}$. We may write $m = 1 + 4k$ and $n = 2 + 4\ell$, where $k, \ell \in \mathbb{Z}$. But then

$$m + n = 1 + 4k + 2 + 4\ell = (1 + 2) + 4(k + \ell) = 3 + 4j$$

where $j = k + \ell \in \mathbb{Z}$, and

$$mn = (1 + 4k)(2 + 4\ell) = 2 + 4\ell + 8k + 16k\ell = 2 + 4(\ell + 2k + 4k\ell) = 2 + 4i$$

where $i = \ell + 2k + 4k\ell \in \mathbb{Z}$. So $m + n \in 3 + 4\mathbb{Z}$ and $mn \in 2 + 4\mathbb{Z}$. $\square$

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3. Prove that

$$\mathbb{Z} = 4\mathbb{Z} \cup (1 + 4\mathbb{Z}) \cup (2 + 4\mathbb{Z}) \cup (3 + 4\mathbb{Z}).$$

SOLUTION:

1) First, we show that $\mathbb{Z} \subseteq 4\mathbb{Z} \cup (1+4\mathbb{Z}) \cup (2+4\mathbb{Z}) \cup (3+4\mathbb{Z})$: let $m \in \mathbb{Z}$. By the division algorithm, we have $m = 4q + r$ where $q, r \in \mathbb{Z}$ and either $r = 0$, $r = 1$, $r = 2$, or $r = 4$.

In the first case we have $m = 4q$, so $m \in 4\mathbb{Z}$. In the second case we have $m = 4q+1$, so $m \in 1+4\mathbb{Z}$. In the third case we have $m = 4q+2$, so $m \in 2+4\mathbb{Z}$. In the fourth case we have $m = 4q+3$, so $m \in 3+4\mathbb{Z}$. So either $m \in 4\mathbb{Z}$ or $m \in 1+4\mathbb{Z}$ or $m \in 2+4\mathbb{Z}$ or $m \in 3+4\mathbb{Z}$, so by definition of union, $m \in 4\mathbb{Z} \cup (1+4\mathbb{Z}) \cup (2+4\mathbb{Z}) \cup (3+4\mathbb{Z})$.

So $\mathbb{Z} \subseteq 4\mathbb{Z} \cup (1+4\mathbb{Z}) \cup (2+4\mathbb{Z}) \cup (3+4\mathbb{Z})$.

2) Next, we show that $4\mathbb{Z} \cup (1+4\mathbb{Z}) \cup (2+4\mathbb{Z}) \cup (3+4\mathbb{Z}) \subseteq \mathbb{Z}$: let $m \in 4\mathbb{Z} \cup (1+4\mathbb{Z}) \cup (2+4\mathbb{Z}) \cup (3+4\mathbb{Z})$. Then $m \in 4\mathbb{Z}$ or $m \in 1+4\mathbb{Z}$ or $m \in 2+4\mathbb{Z}$ or $m \in 3+4\mathbb{Z}$. In each case, $m \in \mathbb{Z}$, since each of these three sets is a set of integers. So $4\mathbb{Z} \cup 4\mathbb{Z}+1 \cup 4\mathbb{Z}+2 \subseteq \mathbb{Z}$.

So $4\mathbb{Z} \cup (1+4\mathbb{Z}) \cup (2+4\mathbb{Z}) \cup (3+4\mathbb{Z}) \subseteq \mathbb{Z}$.

Therefore, $\mathbb{Z} = 4\mathbb{Z} \cup (1+4\mathbb{Z}) \cup (2+4\mathbb{Z}) \cup (3+4\mathbb{Z})$. □

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4. Let $x, y, z \in \mathbb{Z}$. Use the principle of mathematical induction to prove that, for all $n \in \mathbb{N}$,

$$z|(x-y) \Rightarrow z|(x^n - y^n).$$

Hint: you may use the fact that, for $x, y \in \mathbb{Z}$ and $k \in \mathbb{N}$,

$$x^{k+1} - y^{k+1} = x(x^k - y^k) + y^k(x-y).$$

Please clearly identify your base step, induction hypothesis, inductive step, and the conclusion of your proof.

SOLUTION:

Let $x, y \in \mathbb{Z}$, and let $A(n)$ be the statement in question.

Step 1: Is $A(1)$ true? Yes, because if $z|(x-y)$, then certainly $z|(x^1 - y^1)$.

Step 2: Assume $A(k)$:

$$z|(x-y) \Rightarrow z|(x^k - y^k).$$

Now suppose $z|(x-y)$. Note that

$$x^{k+1} - y^{k+1} = x(x^k - y^k) + y^k(x-y).$$

But $z|(x^k - y^k)$ by the induction hypothesis, and $z|(x-y)$ by assumption. So by Exercise B(i)-3 in S-POP, $z|\big(x(x^k - y^k) + y^k(x-y)\big)$, so $z|(x^{k+1} - y^{k+1})$. So $A(k+1)$ follows.

So, by the principle of mathematical induction, $A(n)$ is true for all $n \in \mathbb{N}$. □

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5. Use the principle of mathematical induction to prove the following:

Proposition.

For any natural number $n \geq 3$,

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n-1}{2} = \binom{n}{3}.$$

Please clearly identify your base step, induction hypothesis, inductive step, and the conclusion of your proof. Hint: you should start at $n = 3$ instead of $n = 1$. Also, you may want to use the following formula, which we proved in class:

$$\binom{k}{j} + \binom{k}{j-1} = \binom{k+1}{j}, \tag{$*$}$$

for $j, k \in \mathbb{Z}$ and $1 \leq j \leq k$. (You DON'T need to prove $(*)$.)

SOLUTION:

Let $A(n)$ be the statement in question.

Step 1: Is $A(3)$ true?

$$\binom{2}{2} = \binom{3}{3}$$

(since $1 = 1$), so $A(3)$ is true.

Step 2: Assume

$$A(k): \quad \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{k-1}{2} = \binom{k}{3}.$$

Then

$$\begin{aligned} &\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{k}{2} \\ &= \left(\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{k-1}{2}\right) + \binom{k}{2} \\ &= \binom{k}{3} + \binom{k}{2} = \binom{k+1}{3}, \end{aligned}$$

the last step by $(*)$. So $A(k) \Rightarrow A(k+1)$.

So, by the principle of mathematical induction, $A(n)$ is true for all $n \geq 3$. $\square$

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6. Imagine flipping an unfair coin, for which $P(\text{heads}) = 1/4 = 0.25$, six times. (We'll assume that heads and tails are the only possible outcomes. That is, the coin can't land on its edge, hover in midair forever, etc. Also, we'll assume that no flip affects any other.)

(a) How many possible outcomes are there? (Think of an outcome as being a list, of length 6, of H's and T's – e.g. HTHHTT – where the first letter in the list designates how the coin landed the first time, and so on).

SOLUTION: There are $2^6 = 64$ possible outcomes.

(b) How many possible outcomes are there in which the first *or* the fourth coin lands heads (or both)? There are no restrictions here on how the other four coins might land.

SOLUTION: There are 2^5 possible outcomes where the first lands heads, 2^5 possible outcomes where the fourth lands heads, and 2^4 possible outcomes where they both do. So, by the inclusion-exclusion principle, there are

$$2^5 + 2^5 - 2^4 = 48$$

possible outcomes where one or the other lands heads (or both do).

Another way of counting is by the subtraction principle. The number of possible outcomes in which the first *or* the fourth coin lands heads (or both) is the total number of outcomes, which is 64, minus the number in which both the first and fourth land tails, which is 16. And $64 - 16 = 48$.

(c) How many outcomes have exactly two heads, which are consecutive (right next to each other)? Please explain.

SOLUTION: Either the first two are heads and the rest are tails, or the second and third are heads and the rest are tails, or the third and fourth are heads and the rest are tails, or the fourth and fifth are heads and the rest are tails, or the fifth and sixth are heads are heads and the rest are tails. There's only one way for each of these events to happen. So 5 outcomes have exactly two heads, which are consecutive.

(d) What's more likely: that the first flip lands heads and the second flip lands tails (with no restrictions on what the other flips do: they could land heads or tails), or that exactly three of the flips land heads? Please explain.

SOLUTION: We have

$$P(\text{the first flip lands heads and the second flip lands tails}) = 0.25 \cdot 0.75 = 0.1875.$$

On the other hand, if X denotes the number that land heads, then

$$P(X = 3) = \binom{6}{3} \cdot 0.25^3 \cdot 0.75^3 = 0.131836.$$

So it's more likely that the first flip lands heads and the second flip lands tails.

(e) What's more likely: that an even number of flips land heads, or that an odd number land heads? Please explain.

SOLUTION: The probability that an even number land heads is

$$\begin{aligned}&P(X = 0) + P(X = 2) + P(X = 4) + P(X = 6)\\ &= \binom{6}{0} \cdot 0.25^0 \cdot 0.75^6 + \binom{6}{2} \cdot 0.25^2 \cdot 0.75^4 + \binom{6}{4} \cdot 0.25^4 \cdot 0.75^2 + \binom{6}{6} \cdot 0.25^6 \cdot 0.75^0\\ &= 0.507813,\end{aligned}$$

while the probability that an odd number land heads is

$$\begin{aligned}&P(X=1)+P(X=3)+P(X=5)\\&=\binom{6}{1}\cdot 0.25^1\cdot 0.75^5+\binom{6}{3}\cdot 0.25^3\cdot 0.75^3+\binom{6}{5}\cdot 0.25^5\cdot 0.75^1\\&=0.492188.\end{aligned}$$

So it's slightly more likely that an even number land heads.

7. Which is more likely: (a) hitting the jackpot (matching all seven numbers) in a lottery, where seven different numbers are drawn from a set of 48 numbers, or (b), tossing a fair coin 25 (independent) times, and getting all tails?

SOLUTION: There are $\binom{48}{7}$ equally likely ways of choosing seven numbers out of 48, and only one of those choices will exactly match the seven winning numbers. So

$$P(\text{winning the lottery})=\frac{1}{\binom{48}{7}}\approx 1.35816\cdot 10^{-8}.$$

On the other hand, the probability of a fair coin landing heads on a single toss is $1/2$, so the probability of this happening 25 times in a row is

$$P(25\text{ out of }25\text{ tosses land heads})=\left(\frac{1}{2}\right)^{25}\approx 2.98023\cdot 10^{-8}.$$

So having the coin land heads 25 out of 25 times is more likely.

8. (Harder problem.) You wash 14 pairs of socks, each pair a different color from the other pairs, and realize, after taking everything out of the dryer, that 8 socks are lost. So you are left with $(14\cdot 2)-8=20$ socks. Find the probability that:

A. You are left with 10 matching pairs (this is the best case scenario);

B. You are left with 6 matching pairs (this is the worst case scenario).

Some hints: (i) First, what is the total number of ways of losing 8 out of 28 socks? (ii) To be left with 10 matching pairs is to say that 4 of the original 14 pairs were lost. (iii) To be left with 6 matching pairs is to say that 8 socks, all of different colors, were lost. How many ways are there of choosing the 8 colors lost, and for each of these colors, how many ways are there of choosing a sock of that color?

SOLUTION: There are $\binom{28}{8}$ equally likely ways of choosing eight socks from the 28 total socks. How many of these ways yield the best case scenario? Well, to be left with ten matching pairs is to say that four of the original 14 pairs were lost. There are $\binom{14}{4}$ ways of choosing four pairs from the 14 total pairs. So

$$P(\text{best case scenario})=\frac{\binom{14}{4}}{\binom{28}{8}}\approx 0.000322061.$$

Now how many of the $\binom{28}{8}$ ways of choosing eight socks yield the worst case scenario? Well, to be left with six matching pairs is to say that eight of the original pairs were unpaired, meaning the eight socks lost consist of one sock of each color. There are $\binom{14}{8}$ ways of choosing the eight colors, and for each color chosen, there are two ways of choosing a sock of that color. So

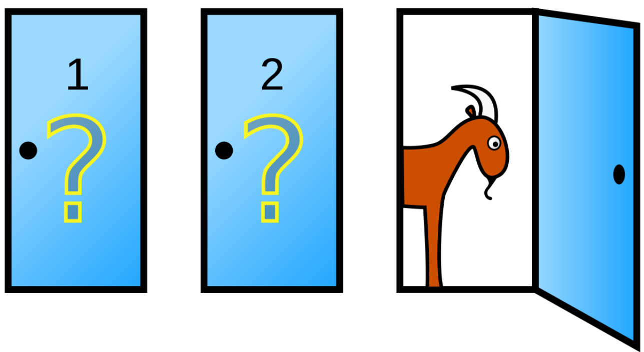
$$P(\text{worst case scenario}) = \frac{\binom{14}{8} \cdot 2^8}{\binom{28}{8}} \approx 0.247343.$$

So $P(\text{worst case scenario}) > P(\text{best case scenario})$ (by a lot).

9. In this problem set, we consider "the Monty Hall Problem," which is a probabilistic curiosity.

Consider this hypothetical scenario from "Let's Make a Deal," a popular TV game show that ran for decades starting in the sixties, hosted by Monty Hall:

> A contestant is shown three doors, and told that there is a new car behind one of them, and a goat behind each of the other two. The contestant chooses a door. Monty Hall **does not** open the chosen door, but **does** reveal a goat behind one of the two doors the contestant did *not* choose.



> Monty Hall then asks the contestant, "Would you like to stick with your first choice, and take whatever lies behind the door you just picked, or instead switch, and go with whatever lies behind the other closed door?"

QUESTION: is the contestant better off switching, or sticking with their original choice? ("Better off" means "more likely to win the car.")

Intuitively, it seems like it shouldn't matter – the contestant has no information about what's behind either of the two closed doors. Right?

Well, let's see. To answer this question, we consider each strategy (STICKING and SWITCHING) separately, and calculate the probability of winning in each case. To this end, let's give names to the three doors: let's call the door with the car behind it W, and call the other two doors L_1 and L_2. Of course, the contestant doesn't know which door is which, though the contestant **does** know, once Monty reveals a goat behind one of the doors, that this door (the one Monty opens) is either door L_1 or door L_2.

Part A: STICKING strategy. This is the strategy where the contestant DOES NOT SWITCH after a goat is revealed behind one of the doors.

Fill in the blanks (there are three of them): Since the contestant is **not** going to switch, the contestant will win if their original pick is the $\underline{\quad W \quad}$ (W, L_1, or L_2) door, and will lose otherwise. Since there are **three** doors, and only **one** of them is the $\underline{\quad W \quad}$ (W, L_1, or L_2) door, the probability that the contestant will win, with this STICKING strategy, is therefore equal to $\underline{\quad 1/3 \quad}$ (a fraction, like 1/2, 4/7, etc.) .

Now we consider

Part B: SWITCHING strategy. This is the strategy where the contestant DOES SWITCH after a goat is revealed behind one of the doors. There are three cases to consider here:

Case (i). Fill in the blank (there is just one): Suppose the door the contestant picks in the first place (before seeing the goat) is the W door. Then, since the contestant **is going to switch**, the contestant will $\underline{\quad \text{lose} \quad}$ (win, lose).

Case (ii). Fill in the blanks (there are three of them): Now suppose the door that the contestant picks in the first place is the L_1 door. Monty then reveals a goat behind one of the remaining two doors, so that door (the one Monty opens) *must* be the $\underline{\quad L_2 \quad}$ (W, L_1, or L_2) door. So the remaining door – the one the contestant is going to switch to – must be the $\underline{\quad W \quad}$ (W, L_1, or L_2) door. CONCLUSION: if the first door the contestant picks is the L_1 door, then by switching, the contestant is guaranteeed to $\underline{\quad \text{win} \quad}$ (win, lose).

Case (iii). Fill in the blanks (there are three of them): Now suppose the door that the contestant picks in the first place is the L_2 door. Monty then reveals a goat behind one of the remaining two doors, so that door (the one Monty opens) *must* be the $\underline{\quad L_1 \quad}$ (W, L_1, or L_2) door. So the remaining door – the one the contestant is going to switch to – must be the $\underline{\quad W \quad}$ (W, L_1, or L_2) door. CONCLUSION: if the first door the contestant picks is the L_2 door, then by switching, the contestant is guaranteed to $\underline{\quad \text{win} \quad}$ (win, lose).

SUMMARY of SWITCHING strategy: there are three doors total, and under the SWITCHING strategy, precisely $\underline{\quad 2 \quad}$ (0, 1, 2, or 3) of these doors will yield a win if chosen in the first place. So the probability of winning, under the SWITCHING strategy, is $\underline{\quad 2/3 \quad}$ (a fraction, like 1/2, 4/7, etc.) .

Part C: Comparison of the two strategies. Fill in the blanks (there are three of them): In **Part A** above we saw that, with the STICKING strategy, the contestant's probability of winning is $\underline{\quad 1/3 \quad}$ (a fraction, like 1/2, 4/7, etc.) . On the other hand, in **Part B** above we saw that, with the SWITCHING strategy, the contestant's probability of winning is

2/3 (a fraction, like 1/2, 4/7, etc.). Therefore the SWITCHING (STICKING, SWITCHING) strategy is the better strategy for winning.

Part D: Reflection. Looking back, can you now explain, intuitively (using as little actual math as possible), why one strategy is better than the other?